Introduction to Mathematical Quantum Theory

Solution to all the Exercises

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1 Exercise Sheet 1

1.1 Exercise 1 - Examples of Fourier transforms

a Consider the function $f \in L^1(\mathbb{T})$ defined as the periodization of

$$f(x) := x(2\pi - x). \tag{1}$$

Calculate the Fourier coefficients of f and use them to prove that

$$\sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{2}$$

b Let σ be a positive real number and $\mathbf{v},\ \mathbf{u} \in \mathbb{R}^d$. Consider the function $g_{\sigma,\mathbf{v},\mathbf{u}}$ in the space $L^2\left(\mathbb{R}^d\right)$ with $d \in \mathbb{N}$ defined as

$$g_{\sigma,\mathbf{v},\mathbf{u}}(\mathbf{x}) := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}. \tag{3}$$

Then prove that $\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}}$, i.e.

$$\mathcal{F}\left[\left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}\right](\mathbf{k}) = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{1}{2\sigma}|\mathbf{k}-\mathbf{u}|^2 - i\mathbf{u}\cdot(\mathbf{k}-\mathbf{v})}. \tag{4}$$

Proof. For the proof of **a**, first consider the coefficients of f; if $k \in \mathbb{Z}\setminus\{0\}$ those are given as

$$\begin{aligned}
\widehat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(2\pi - x) e^{-ikx} dx \\
&= \frac{i}{\sqrt{2\pi}k} \left[x(2\pi - x) e^{-ikx}\right]_0^{2\pi} - \frac{\sqrt{2}i}{\sqrt{\pi}k} \int_0^{2\pi} (\pi - x) e^{-ikx} dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}k^2} \left[(\pi - x) e^{-ikx}\right]_0^{2\pi} + \frac{\sqrt{2}}{\sqrt{\pi}k^2} \int_0^{2\pi} e^{-ikx} dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}k^2} \left[-\pi e^{-2\pi ki} - \pi + \frac{i}{k}\left(e^{-2\pi ki} - 1\right)\right] = -\frac{\sqrt{8\pi}}{k^2}.
\end{aligned}$$

On the other hand when $k=0$ we have

$$\widehat{f}(0)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}x(2\pi-x)\,dx=\frac{1}{\sqrt{2\pi}}\left[\pi x^2-\frac{1}{3}x^3\right]_0^{2\pi}=\frac{\sqrt{8\pi}\pi^2}{3}.$$

We then use the fact that $f(0)=0$ to get

$$\begin{aligned}0=f(0)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)&=\frac{\sqrt{8\pi}\pi^2}{3}-2\sqrt{8\pi}\sum_{k=0}^{+\infty}\frac{1}{k^2}\\ \implies\sum_{k=0}^{+\infty}\frac{1}{k^2}&=\frac{\pi^2}{6},\end{aligned}$$

which concludes the proof of (2).

For the proof of **b**, recall that for any positive real number $\alpha>0$ we have

$$\int_{\mathbb{R}^d}e^{-\alpha|\mathbf{x}|^2}d\mathbf{x}=\left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}}.$$

Consider now the function

$$h_\sigma(\mathbf{x}):=g_{\sigma,\mathbf{0},\mathbf{0}}(\mathbf{x})=\left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}}e^{-\frac{\sigma}{2}|\mathbf{x}|^2}.$$

In general we have that

$$\partial_{x_j}h_\sigma(\mathbf{x})=-\sigma x_j h_\sigma(\mathbf{x}).$$

Consider then the derivative on the j-th component of $\widehat{h}_\sigma$. Now, given that h_σ is an exponentially decaing continuous function, we can apply Leibniz theorem and integration by part to get

$$\begin{aligned}\partial_{k_j}\widehat{h}_\sigma(\mathbf{k})&=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}h_\sigma(\mathbf{x})\,\partial_{k_j}\left(e^{-i\mathbf{k}\cdot\mathbf{x}}\right)d\mathbf{x}\\ &=-i\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}x_j h_\sigma(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\ &=\frac{i}{\sigma}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\partial_{x_j}h_\sigma(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\ &=-\frac{i}{\sigma}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}h_\sigma(\mathbf{x})\,\partial_{x_j}e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\\ &=-\frac{1}{\sigma}k_j\widehat{h}_\sigma(\mathbf{k}).\end{aligned}$$

This is a well defined differential equation, with initial datum

$$\widehat{h}_\sigma(\mathbf{0})=\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}h_\sigma(\mathbf{x})\,d\mathbf{x}=\left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}}$$

If we now suppose that $\widehat{h}_\sigma(\mathbf{k}) = f_1(k_1)\cdot\ldots\cdot f_d(k_d)$, we get that for any j

$$f_j'(t) = -\frac{1}{\sigma} t f_j(t),$$

and therefore, integrating t between 0 and k_j we get

$$-\frac{k_j^2}{2\sigma} = -\int_0^{k_j} \frac{1}{\sigma} t dt = \int_0^{k_j} \frac{f_j'(t)}{f_j(t)} dt = [\log(f_j(t))]_0^{k_j} = \log\left(\frac{f_j(k_j)}{f_j(0)}\right),$$

and therefore we get

$$\begin{aligned} & f_j(k_j) = f_j(0) e^{-\frac{k_j^2}{2\sigma}} \\ \Rightarrow & \widehat{h}_\sigma(\mathbf{k}) = \prod_{j=1}^d \left(f_j(0) e^{-\frac{k_j^2}{2\sigma}}\right) = \widehat{h}_\sigma(\mathbf{0}) e^{-\frac{|\mathbf{k}|^2}{2\sigma}} = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{|\mathbf{k}|^2}{2\sigma}} = h_{\sigma^{-1}}(\mathbf{k}). \end{aligned}$$

Recall now that for any vector $\mathbf{r} \in \mathbb{R}^d$ the operators $T_\mathbf{r}$ and $M_\mathbf{r}$ are defined as

$$T_\mathbf{r} f(\mathbf{x}) := f(\mathbf{x}-\mathbf{r}), \qquad M_\mathbf{r} f(\mathbf{x}) := e^{-i\mathbf{r}\cdot\mathbf{x}} f(\mathbf{x}), \qquad \forall f \in L^1\left(\mathbb{R}^d\right).$$

Then, we saw before that

$$\mathcal{F}T_\mathbf{r} = M_\mathbf{r}\mathcal{F}, \qquad \mathcal{F}M_\mathbf{r} = T_{-\mathbf{r}}\mathcal{F}.$$

We now get to calculate the transform of $g_{\sigma,\mathbf{v},\mathbf{u}}$. First notice that $g_{\sigma,\mathbf{v},\mathbf{u}} = M_{-\mathbf{u}} T_\mathbf{v} g_{\sigma,\mathbf{0},\mathbf{0}} = M_{-\mathbf{u}} T_\mathbf{v} h_\sigma$. Notice now that for any $f \in L^2\left(\mathbb{R}^d\right)$ we get

$$(T_\mathbf{u} M_\mathbf{v} f)(\mathbf{x}) = (M_\mathbf{v} f)(\mathbf{x}-\mathbf{u}) = e^{-i\mathbf{v}\cdot(\mathbf{x}-\mathbf{u})} f(\mathbf{x}-\mathbf{u}) = e^{i\mathbf{v}\cdot\mathbf{u}} (M_\mathbf{v} T_\mathbf{u} f)(\mathbf{x}).$$

We then have

$$\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = \mathcal{F}M_{-\mathbf{u}} T_\mathbf{v} h_\sigma = T_\mathbf{u} M_\mathbf{v} h_{\sigma^{-1}} = e^{i\mathbf{v}\cdot\mathbf{u}} M_\mathbf{v} T_\mathbf{u} h_{\sigma^{-1}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}},$$

which concludes the proof.

□

1.2 Exercise 2 - Properties of operator norm and definition of boundedness (complement to the exercise session)

Consider V_1 and V_2 two normed vector spaces over[1] $\mathbb{F}$ and $T : V_1 \to V_2$ a linear mapping. Define $\|T\|_{V_1,V_2}$ as

$$\|T\| := \sup_{v\in V_1,\ v\neq 0} \frac{\|Tv\|}{\|v\|}. \tag{5}$$

[1]Here and in the following $\mathbb{F}$ can be chosen to be either $\mathbb{R}$ or $\mathbb{C}$.

For a generic linear mapping T we have $\|T\| \in [0, +\infty]$. Prove that

$$\|T\| = \sup_{v\in V_1,\ \|v\|_{V_1}=1} \|Tv\| \tag{6}$$
$$= \sup_{v\in V_1,\ \|v\|_{V_1}\leqslant 1} \|Tv\|. \tag{7}$$

Prove moreover that the following are equivalent

- **a** T is continuous.
- **b** T is continuous in 0, meaning that for any sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$,

$$v_n \to 0 \implies Tx_n \to 0. \tag{8}$$

- **c** The quantity $\|T\|$ is finite, meaning that $\|T\| < +\infty$.

Proof. To prove (6) we get

$$\|T\| = \sup_{v\in V_1,\ v\neq 0} \frac{\|Tv\|}{\|v\|} = \sup_{v\in V_1,\ v\neq 0} \left\| T\left(\frac{v}{\|v\|}\right)\right\|$$
$$= \sup_{v\in V_1,\ \|v\|=1} \|Tv\|.$$

To prove (7) first notice that given that the unit sphere is a subset of the corresponding unit ball we have

$$\sup_{v\in V_1,\ \|v\|=1} \|Tv\| \leqslant \sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\|.$$

On the other hand, suppose that $v \in V_1$ with $\|v\| \leqslant 1$, then

$$\sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\| \leqslant \sup_{v\in V_1,\ \|v\|\leqslant 1} \|T\| \|v\| = \|T\|,$$

which concludes the proof of the first part of the exercise.

Next notice that **a** implies **b** trivially.

To prove that **b** implies **c**, we have that if T is continuous, the preimage of any open set is open. In particular, consider[2] $T^{-1}(B_1(0))$. Given that $0 \in T^{-1}(B_1(0))$ and T is continuous, there exists a positive real number R such that $B_R(0) \subseteq T^{-1}(B_1(0))$, or equivalently, by linearity of T, that $T(B_1(0)) \subseteq B_{R^{-1}}(0)$. This can be also written as

$$\|v\| \leqslant 1 \Longrightarrow \|Tv\| \leqslant \frac{1}{R}$$

and implies in particular that

$$\|T\| = \sup_{v\in V_1,\ \|v\|\leqslant 1} \|Tv\| \leqslant \frac{1}{R},$$

[2] Recall that $B_r(v)$ denote the ball of radius r around v.

which implies **c**.

To prove that **c** implies **a**, consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$ such that $v_n \to v$ in V_1. Then we have

$$\|Tv_n - Tv\| = \|T(v_n - v)\| \leqslant \|T\| \|v_n - v\| \to 0,$$

completing the proof of the exercise.

□

1.3 Exercise 3 - Young Inequality

Consider $p,\ q,\ r \in [1, +\infty]$ such that

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}. \tag{9}$$

Let $f \in L^q\left(\mathbb{R}^d\right)$, $g \in L^r\left(\mathbb{R}^d\right)$; prove that

$$\|f * g\|_p \leqslant \|f\|_q \|g\|_r . \tag{10}$$

Hint: Consider the functions $\alpha,\ \beta,\ \gamma$ defined as

$$\alpha(\mathbf{x},\mathbf{y}) := |f(\mathbf{y})|^q |g(\mathbf{x}-\mathbf{y})|^r , \tag{11}$$

$$\beta(\mathbf{y}) := |f(\mathbf{y})|^q , \tag{12}$$

$$\gamma(\mathbf{x},\mathbf{y}) := |g(\mathbf{x}-\mathbf{y})|^r , \tag{13}$$

notice that

$$|f * g(\mathbf{x})| \leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x},\mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x},\mathbf{y})^{\frac{p-r}{pr}} \, d\mathbf{y} \tag{14}$$

and that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = 1 \tag{15}$$

to apply Hölder inequality.

Proof. Consider $\alpha,\ \beta$ and γ as in the Hint. From basic algebraic properties of the Hölder conjugate exponents we get that

$$\alpha(\mathbf{x},\mathbf{y}) \beta(\mathbf{y}) \gamma(\mathbf{x},\mathbf{y}) = |f(\mathbf{y}) g(\mathbf{x}-\mathbf{y})| .$$

Given that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = \frac{1}{q} + \frac{1}{r} - \frac{1}{p} = 1,$$

applying the previous equality to (10) and using Hölder inequality we get

$$\begin{aligned} |f * g(\mathbf{x})| &\leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x},\mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x},\mathbf{y})^{\frac{p-r}{pr}} \, d\mathbf{y} \\ &\leqslant \left\| \alpha(\mathbf{x},\cdot)^{\frac{1}{p}} \right\|_p \left\| \beta^{\frac{p-q}{pq}} \right\|_{\frac{pq}{p-q}} \left\| \gamma(\mathbf{x},\cdot)^{\frac{p-r}{pr}} \right\|_{\frac{pr}{p-r}} \\ &= \|\alpha(\mathbf{x},\cdot)\|_1^{\frac{1}{p}} \|f\|_q^{\frac{p-q}{p}} \|g\|_r^{\frac{p-r}{p}} . \end{aligned}$$

Now expanding the norm of α we get that

$$\begin{aligned}\|\alpha\|_1 &= \int_{\mathbb{R}^{2d}} |f(\mathbf{y})|^q \, |g(\mathbf{x}-\mathbf{y})|^r \, d\mathbf{x} d\mathbf{y} \\ &= \|f\|_q^q \, \|g\|_r^r \, .\end{aligned}$$

So now we get

$$\begin{aligned}\|f * g\|_p &= \left[\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} f(\mathbf{y})\, g(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\right]^p d\mathbf{y}\right]^{\frac{1}{p}} \leqslant \|f\|_q^{\frac{p-q}{p}} \, \|g\|_r^{\frac{p-r}{p}} \left[\int_{\mathbb{R}^d} \|\alpha(\mathbf{x},\cdot)\|_1 \, d\mathbf{y}\right]^{\frac{1}{p}} \\ &= \|f\|_q^{\frac{p-q}{p}} \, \|g\|_r^{\frac{p-r}{p}} \, \|g\|_r^{\frac{r}{p}} \, \|f\|^{\frac{q}{p}} = \|f\|_q \, \|g\|_r \, ,\end{aligned}$$

which concludes our proof.

□

1.4 Exercise 4 - Fourier transform and sinc

a Prove that there exists a positive real number C such that we have

$$\sup_{0 \leqslant a < b < +\infty} \left| \int_a^b \frac{\sin x}{x} dx \right| \leqslant C. \tag{16}$$

Hint: Consider the function

$$F(t) := \int_0^\eta e^{-tx} \frac{\sin x}{x} dx. \tag{17}$$

Deduce a bound on $F'(t)$ uniform in η. Apply the fundamental theorem of calculus for $F(0)$ to conclude.

b Consider an odd function $f \in L^1(\mathbb{R})$. Prove that for any such function we have

$$\sup_{0 \leqslant a < b < +\infty} \left| \int_a^b \frac{\widehat{f}(k)}{k} dk \right| \leqslant \frac{C}{(2\pi)^{\frac{d}{2}}} \|f\|_1 \, . \tag{18}$$

c Let $g(k)$ be a continuous odd function on the line such that is equal to $1/\log k$ for any $k \geqslant 2$. Prove that there cannot be an $L^1(\mathbb{R})$ function whose Fourier transform is g.

Proof. We first prove **a**; given that the function sinc is even, it is enough to bound the following quantity:

$$\left| \int_0^\eta \frac{\sin x}{x} dx \right|,$$

with η a positive real number.

Consider now the function $F(t)$ defined as

$$F(t) := \int_0^\eta e^{-tx} \frac{\sin x}{x} dx.$$

Then $F(t)$ is well defined and continuous for any real number t and we have that $F(0)$ is our initial quantity and $F(t) \to 0$ as $t \to +\infty$. Moreover the derivative of F gives

$$\begin{aligned} F'(t) &= \int_0^\eta e^{-tx} \sin x \, dx = -\mathrm{Im}\left(\int_0^\eta e^{-(t+i)x} dx\right) \\ &= \frac{1}{1+t^2}\left(1 - te^{-\eta t}\sin\eta - e^{-\eta t}\cos\eta\right). \end{aligned}$$

Using now the fundamental theorem of calculus we get

$$F(0) = F(T) + \int_T^0 F'(t)\, dt$$

for any positive T and hence, taking the limit $T \to +\infty$

$$\begin{aligned} F(0) &= \lim_{T\to+\infty} F(0) \\ &= \lim_{T\to+\infty}\left[F(T) - \int_0^T F'(t)\, dt\right] \\ &= -\int_0^{+\infty} F'(t)\, dt \\ &= \int_0^{+\infty} \frac{1}{1+t^2}\left(te^{-\eta t}\sin\eta + e^{-\eta t}\cos\eta - 1\right) dt. \end{aligned}$$

For any positive real number η we get that

$$\begin{aligned} \sup_{t>0}\left|te^{-\eta t}\sin\eta\right| &= \left|\frac{\sin\eta}{\eta}\right| \sup_{t>0}\ te^{-t} = \left|\frac{\sin\eta}{\eta e}\right| \leqslant e^{-1} \\ \sup_{t>0}\left|e^{-\eta t}\cos\eta - 1\right| &= \sup_{t>0}\left(1 - e^{-t}\cos\eta\right) = 1 \end{aligned}$$

and therefore we can bound $|F(0)|$ as

$$\begin{aligned} \left|\int_0^\eta e^{-tx}\frac{\sin x}{x} dx\right| = |F(0)| &\leqslant \frac{1+e}{e}\int_0^{+\infty}\frac{1}{1+t^2} dt \\ &= \frac{\pi(1+e)}{2e}. \end{aligned}$$

Next, to prove **b** consider f an odd function. Then we have

$$f(x) = -f(-x) \quad \Longrightarrow \quad f(x) = \frac{1}{2}\left(f(x) - f(-x)\right).$$

This implies that if we consider the Fourier transform of f we get

$$\begin{aligned} \widehat{f}(k) &= \frac{1}{2(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}} \left[f(x) - f(-x)\right] e^{-ikx} dx \\ &= \frac{1}{2(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}} f(x)\left[e^{-ikx} - e^{ikx}\right] dx \\ &= \frac{i}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}} f(x)\sin(kx)\, dx \\ &= \frac{2i}{(2\pi)^{\frac{d}{2}}}\int_0^{+\infty} f(x)\sin(kx)\, dx. \end{aligned}$$

We substitute this in (18) to get

$$\begin{aligned}\left|\int_a^b \frac{\widehat{f}(k)}{k} dk\right| &= \frac{2}{(2\pi)^{\frac{d}{2}}} \left|\int_a^b \int_0^{+\infty} f(x) \frac{\sin(kx)}{k} dx dk\right| \\ &\leqslant \frac{2}{(2\pi)^{\frac{d}{2}}} \int_0^{+\infty} |f(x)| \int_a^b \left|\frac{\sin(kx)}{k}\right| dk dx \\ &= \frac{2}{(2\pi)^{\frac{d}{2}}} \int_0^{+\infty} |f(x)| \int_{xa}^{xb} \left|\frac{\sin\ k}{k}\right| dk dx \\ &\leqslant \frac{2\Xi}{(2\pi)^{\frac{d}{2}}} \int_0^{+\infty} |f(x)| \, dx = \frac{\Xi}{(2\pi)^{\frac{d}{2}}} \|f\|_1\end{aligned}$$

where in the last inequality we used (16). This concludes the proof of **b**.

To prove **c** now, suppose $g = \widehat{h}$. Then on one hand from (18) for any positive real number $R > 2$ we would have

$$\left|\int_2^R \frac{g(k)}{k} dk\right| = \left|\int_2^R \frac{\widehat{h}(k)}{k} dk\right| \leqslant \frac{\Xi}{(2\pi)^{\frac{d}{2}}} \|h\|_1 .$$

On the other hand, we get that

$$\left|\int_2^R \frac{g(k)}{k} dk\right| = \int_2^R \frac{1}{k \log k} dk = \int_{\log 2}^{\log R} \frac{1}{z} dz = \log \frac{\log R}{\log 2},$$

where in the second equality we used the change of variables $z = \log k$. Now the last term goes to infinity as R goes to infinity, but this is absurd given that we proved above that it should be bounded uniformly in R. Therefore such an h does not exists and the proof is complete.

□

2 Exercise Sheet 2

2.1 Exercise 1 - Fourier transform and convolution

Let $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Recall that in class we proved

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}}\,\widehat{f}\widehat{g}. \tag{19}$$

Prove that

$$\widehat{f} * \widehat{g} = (2\pi)^{\frac{d}{2}}\,\widehat{fg}. \tag{20}$$

Hint: Consider the equivalent statement of (19) *for the inverse of the Fourier transform and apply it to* $\widehat{fg}$.

Proof. Recall that the inverse Fourier transform is such that $\check{f}(\mathbf{x}) = \widehat{f}(-\mathbf{x})$. We then use (19) to get

$$\widecheck{f * g}(\mathbf{x}) = \widehat{f * g}(-\mathbf{x}) = (2\pi)^{\frac{d}{2}}\,\widehat{f}(-\mathbf{x})\,\widehat{g}(-\mathbf{x}) = (2\pi)^{\frac{d}{2}}\,\check{f}(\mathbf{x})\,\check{g}(\mathbf{x}).$$

To prove (20), consider $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Then we know that

$$\check{\widehat{f}} = f, \qquad \check{\widehat{g}} = g.$$

Given that $fg \in \mathscr{S}\left(\mathbb{R}^d\right)$, using the formula we deduced for the inverse Fourier transform, we can get

$$\widehat{fg} = \widehat{\check{\widehat{f}}\check{\widehat{g}}} = (2\pi)^{-\frac{d}{2}}\,\widehat{\widecheck{\widehat{f} * \widehat{g}}} = (2\pi)^{-\frac{d}{2}}\,\widehat{f} * \widehat{g}.$$

□

2.2 Exercise 2 - Unique projector (complement to the class)

Let $\mathcal{H}$ be an Hilbert space and V a closed linear subspace of $\mathcal{H}$.

a In class we proved that for any $f \in \mathcal{H}$ there exists an element $g_f \in V$ such that

$$\|f - g_f\| = \min_{h \in V} \|f - h\|. \tag{21}$$

Prove that g_f is the unique element of V that satisfies the minimum.

b In class we proved that g_f is such that $f - g_f \in V^{\perp}$. Prove that there is no other element $h \in V$ such that $f - h \in V^{\perp}$.

Proof. Recall the parallelogram law; for any $f,\ g \in \mathcal{H}$ we have

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2 .$$

To prove **a**, consider $g' \in$ such that

$$\|f-g'\| = \min_{h\in V} \|f-h\| .$$

This implies in particular that $\|f-g'\| = \|f-g_f\|$. By parallelogram law we deduce that

$$\begin{aligned} \|g_f - g'\|^2 &= \left\|(f-g') - (f-g_f)\right\|^2 \\ &= 2\|f-g'\|^2 + 2\|f-g_f\|^2 - \left\|(f-g') + (f-g_f)\right\|^2 \\ &= 4\|f-g_f\|^2 - \left\|2f - (g'+g_f)\right\|^2 = 4\|f-g_f\|^2 - 4\left\|f - \frac{1}{2}(g'+g_f)\right\|^2 . \end{aligned}$$

Given that V is a vector space, we get that $\frac{1}{2}(g'+g_f) \in V$, and therefore

$$\|g_f - g'\|^2 \leqslant 4\|f-g_f\|^2 - 4\inf_{h\in V}\|f-h\|^2 = 0$$

and therefore $g' = g_f$.

To prove **b** suppose that $g' \in V$ is such that $f - g' \in V^\perp$. Then by definition of $V^\perp$, for any $h \in V$ we get

$$\langle h, g_f - g'\rangle = \langle h, f-g'\rangle - \langle h, f-g_f\rangle = 0,$$

where the last equality comes from the fact that both $f-g_f$ and $f-g'$ are in $V^\perp$. We therefore have that $g_f - g' \in V^\perp$. At the same time, $g_f - g' \in V$, and this implies $g' = g_f$.

□

2.3 Exercise 3 - Hilbert space basis with Hahn-Banach

Let $\mathcal{H}$ be an Hilbert space. Prove that there exists a basis for $\mathcal{H}$. Prove moreover that $\mathcal{H}$ is separable if and only if there exists a countable base for it.

Hint: For the first part apply Zorn's Lemma to the set of (also infinite) orthonormal systems ordered by inclusion. Prove that any maximal orthonormal system is a base, i.e. is dense.

For the second part prove and use the following fact: if f is an element of $\mathcal{H}$ and S is a basis for $\mathcal{H}$, there exists a sequence of elements $\{e_n\}_{n\in\mathbb{N}} \subseteq S$ such that $f \in \overline{\operatorname{span}_{\mathbb{K}}\{e_n\}_{n\in\mathbb{N}}}$.

Proof. To prove the first part, call $\mathcal{A}$ the set of all orthonormal systems in $\mathcal{H}$, i.e.

$$\mathcal{A} := \left\{S \subseteq \mathcal{H} :\ \langle\psi,\psi'\rangle = \delta_{\psi,\psi'}\ \forall \psi,\psi' \in S\right\},$$

where $\delta_{\psi,\psi'}$ is 1 if $\psi = \psi'$ and 0 otherwise.

Consider then the set $\mathcal{A}$ with the partial order given by the inclusion. To apply Zorn's Lemma consider $\mathcal{B}$ an inductive ordered subset of $\mathcal{A}$. Consider moreover

$$S_{\mathcal{B}} := \bigcup_{S \in \mathcal{B}} S.$$

We want to prove that this is an upper bound for $\mathcal{B}$.

First we prove that $S_{\mathcal{B}} \in \mathcal{A}$. Given that $\mathcal{H}$ is closed with respect to unions, $S_{\mathcal{B}} \subseteq \mathcal{H}$. Let now $\psi,\ \psi' \in S_{\mathcal{B}}$; then there exist two orthonormal systems $S,\ S'$ such that $\psi \in S \in \mathcal{B}$ and $\psi' \in S' \in \mathcal{B}$; given that $\mathcal{B}$ is ordered, either $S \subseteq S'$ or $S' \subseteq S$. Suppose $S \subseteq S'$; then $\psi, \psi' \in S'$ and we get that $\langle \psi, \psi' \rangle = \delta_{\psi,\psi'}$, and therefore $S_{\mathcal{B}} \in \mathcal{A}$. Given that for any $S \in \mathcal{B}$ we have $S \subseteq S_{\mathcal{B}}$, this is clearly an upper bound for $\mathcal{B}$.

We can now apply Zorn's Lemma to deduce the existence of a maximal element of $\mathcal{A}$. What is left to prove is that this maximal element is a basis. Call S the maximal element; by definition this is an orthonormal system. We prove that it is dense. Suppose it is not; then[3] $V := \overline{\operatorname{span}_{\mathbb{K}}\{S\}}$ is a well defined closed vector space such that $\mathcal{H} \backslash V \neq \varnothing$ and $V^{\perp}$ is nonempty. Let now $\phi \in V^{\perp}$ such that $\|\phi\| = 1$ and let $S_{\phi} := \{\phi\} \cup S$. Clearly $S_{\phi} \subseteq \mathcal{H}$; consider $\psi,\ \psi' \in S_{\phi}$; if $\psi \neq \phi \neq \psi'$ from the fact that S is an orthonormal system we already know that $\langle \psi, \psi' \rangle = \delta_{\psi,\psi'}$. Suppose now $\psi \in S$; given that $\phi \in V^{\perp}$ we get $\langle \psi, \phi \rangle = 0$. Given that $\langle \phi, \phi \rangle = \|\phi\|^2 = 1$ we deduce that S_{ψ} is an orthonormal system. But now $S_{\psi} \supseteq S$ and $S_{\psi} \neq S$, which contradicts the maximality of S. Therefore S is a basis for $\mathcal{H}$.

To prove the second part, we first prove the fact in the hint. Indeed, $f \in \mathcal{H}$ implies that there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \to f$ and

$$f_n = \sum_{j=1}^{N(n)} a_{j,n} e_{j,n}$$

for some $N(n) \in \mathbb{N}$, $\{a_{j,n}\}_{j,n \in \mathbb{N}} \subseteq \mathbb{K}$ and $\{e_{j,n}\}_{j,n \in \mathbb{N}} \subseteq S$. Given that the latter is a countable sequence in S this proves the fact.

Now, we use this fact to prove our Exercise; suppose that $\mathcal{H}$ is separable; therefore, there exists D a dense subset of $\mathcal{H}$ which is countable, i.e., $D = \{d_n\}_{n \in \mathbb{N}}$. But for every $n \in \mathbb{N}$, d_n is in the span of S_n a countable subset of S; we then get the following chain of inequalities:

$$\mathcal{H} = \overline{D} = \overline{\{d_n\}_{n \in \mathbb{N}}} \subseteq \overline{\bigcup_{n \in \mathbb{N}} S_n} \subseteq \overline{S} = \mathcal{H},$$

and the inequalities are in fact equalities.

From this we get that $\bigcup_{n \in \mathbb{N}} S_n$ is dense in $\mathcal{H}$ and given that $\bigcup_{n \in \mathbb{N}} S_n \subset S$, this is also an orthonormal system, therefore is a basis. Moreover, it is union of countable sets, so it is also countable, and this proves the first implication.

Suppose now that S is a countable basis for $\mathcal{H}$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$ and that $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$. Call then $\mathbb{F}$ a countable dense subset of $\mathbb{K}$ We have that $D := \operatorname{span}_{\mathbb{F}} \{S\}$ is countable and dense in $\mathcal{H}$.

□

[3]We indicate with $\operatorname{span}_{\mathbb{K}}\{A\}$ the set of finite linear combinations of elements in A with coefficients in $\mathbb{K}$.

2.4 Exercise 4 - Property of the adjoint (bounded operators)

Let $A,\ B$ bounded operators on an Hilbert space $\mathcal{H}$ and $\alpha,\ \beta \in \mathbb{C}$. Prove the following equalities:

$$\mathrm{id}^* = \mathrm{id} \tag{22}$$
$$(A^*)^* = A \tag{23}$$
$$(AB)^* = B^* A^* \tag{24}$$
$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*. \tag{25}$$

Moreover, prove that A^* is bounded and that $\|A^*\| = \|A\|$.

Proof. For the proof of (22) consider $\psi \in \mathcal{H}$. For any $\phi \in \mathcal{H}$ from the definition of the adjoint we get

$$\langle \phi, \mathrm{id}^* \psi \rangle = \langle \mathrm{id}\, \phi, \psi \rangle = \langle \phi, \psi \rangle \Rightarrow \langle \phi, \mathrm{id}^* \psi - \psi \rangle = 0,$$

and by density we can imply that $\mathrm{id}^* \psi = \psi$.

For the proof of (23) we get that for any $\phi,\ \psi \in \mathcal{H}$

$$\langle \phi, (A^*)^* \psi \rangle = \langle A^* \phi, \psi \rangle = \overline{\langle \psi, A^* \phi \rangle} = \overline{\langle A\psi, \phi \rangle} = \langle \phi, A\psi \rangle.$$

Analogously as before we conclude by density that $(A^*)^* = A$.

For the proof of (24) we get that for any $\phi,\ \psi \in \mathcal{H}$

$$\langle \phi, (AB)^* \psi \rangle = \langle AB\phi, \psi \rangle = \langle B\phi, A^*\psi \rangle = \langle \phi, B^* A^* \psi \rangle.$$

For the proof of (25) we get that for any $\phi,\ \psi \in \mathcal{H}$

$$\begin{aligned} \langle \phi, (\alpha A + \beta B)^* \psi \rangle &= \langle (\alpha A + \beta B)\, \phi, \psi \rangle = \overline{\alpha} \langle A\phi, \psi \rangle + \overline{\beta} \langle B\phi, \psi \rangle \\ &= \overline{\alpha} \langle \phi, A^* \psi \rangle + \overline{\beta} \langle \phi, B^* \psi \rangle = \langle \phi, \left(\overline{\alpha} A^* + \overline{\beta} B^* \right) \psi \rangle, \end{aligned}$$

and we can conclude again by density.

To see that A^* is bounded consider $\psi \in \mathcal{H}$; then we have

$$\|A^*\psi\|^2 = \langle A^*\psi, A^*\psi \rangle = \langle \psi, AA^*\psi \rangle \leqslant \|\psi\| \, \|AA^*\psi\| \leqslant \|\psi\| \, \|A\| \, \|A^*\psi\| ,$$

and therefore we get $\|A^*\psi\| \leqslant \|A\| \, \|\psi\|$; as a consequence we get $\|A^*\| \leqslant \|A\|$, and therefore

$$\|A^*\| \leqslant \|A\| = \left\|(A^*)^*\right\| \leqslant \|A^*\| ,$$

and hence $\|A^*\| = \|A\|$.

□

3 Exercise Sheet 3

3.1 Exercise 1 - Properties of orthogonal projectors

Let $\mathcal{H}$ be a Hilbert space. Let V any closed subspace of $\mathcal{H}$; recall the definition of $V^{\perp}$ as

$$V^{\perp} := \{ f \in \mathcal{H} \mid \langle g, f \rangle = 0 \ \forall g \in V \} . \tag{26}$$

We saw in class that the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = V \oplus V^{\perp}$, meaning that $V \cap V^{\perp} = \{0\}$ and that for any non-zero $f \in \mathcal{H}$ there exists a unique element $f_V \in V$ such that $f - f_V \in V^{\perp}$. Define $P_V f := f_V$; from the uniqueness of f_V this is a well defined linear mapping.

- **a** Prove that $P_V^2 = P_V = P_V^*$.
- **b** Use **a** to prove that P_V is bounded and if $V \neq \{0\}$ then $\|P_V\| = 1$.
- **c** Prove that if V_1 and V_2 are two closed subspaces of $\mathcal{H}$ then[4]

$$V_1 \perp V_2 \qquad \Longleftrightarrow \qquad P_{V_1} P_{V_2} = 0. \tag{27}$$

Proof. We first prove that $P_V^2 = P_V$. To prove this is enough to notice that if $f \in V$ then $P_V f = f$. Indeed, let $g := f - P_V f$. Then by definition $g \in V^{\perp}$. On the other hand, both f and $P_V f$ are in V, therefore $g \in V \cap V^{\perp} = \{0\}$ and this implies $P_V f = f$. Now from the fact that $P_V f \in V$ for any $f \in \mathcal{H}$ we conclude that $P_V^2 f = P_V f$.

To prove that $P_V^* = P_V$, first notice that we have the trivial identity $\mathrm{id} = P_V + (\mathrm{id} - P_V)$. Moreover, by definition of P_V and from the decomposition $\mathcal{H} = V \oplus V^{\perp}$ we get that $(\mathrm{id} - P_V)(\mathcal{H}) \subseteq V^{\perp}$. Consider now $f, \ g \in \mathcal{H}$. We then have

$$\begin{aligned}
\langle g, P_V^* f \rangle &= \langle P_V g, f \rangle \\
&= \langle P_V g, P_V f \rangle + \langle P_V g, (\mathrm{id} - P_V) f \rangle \\
&= \langle P_V g, P_V f \rangle \\
&= \langle g, P_V f \rangle - \langle (\mathrm{id} - P_V) g, P_V f \rangle \\
&= \langle g, P_V f \rangle.
\end{aligned}$$

From the fact that this is true for every $f, \ g \in \mathcal{H}$ we get that $P_V^* = P_V$.

To prove **b** for any $f \in \mathcal{H}$ we get that

$$\begin{aligned}
\|f\|^2 &= \langle f, f \rangle \\
&= \langle P_V f, f \rangle + \langle (\mathrm{id} - P_V) f, f \rangle \\
&= \langle P_V f, P_V f \rangle + \langle (\mathrm{id} - P_V) f, (\mathrm{id} - P_V) f \rangle \\
&= \|P_V f\|^2 + \|(\mathrm{id} - P_V) f\|^2 .
\end{aligned}$$

[4]We denote with $\perp$ the condition of two subspaces of an Hilbert space $\mathcal{H}$ of being orthogonal, i.e., V_1 is orthogonal to V_2, or $V_1 \perp V_2$ if and only if for any $(f, g) \in V_1 \times V_2$ we have $\langle f, g \rangle = 0$.

From this we can deduce that P_V is bounded and that $\|P_V\| \leqslant 1$. If V is non empty, let $f \in V$, $\|f\| = 1$; then $\|P_V f\| = \|f\| = 1$ and this implies that $\|P_V\| = 1$.

To prove **c** first suppose $V_1 \perp V_2$ and $f \in V_2$. By definition of P_{V_1} we have that $f - P_{V_1} f \in V_1^{\perp}$; then we get that

$$P_{V_1} f = f - (f - P_{V_1} f) \in V_1 \cap V_1^{\perp} \Rightarrow P_{V_1} f = 0.$$

Consider now $f \in \mathcal{H}$; given that $P_{V_2} f \in V_2$ we can deduce that $P_{V_1} P_{V_2} = 0$.

Suppose now that $P_{V_1} P_{V_2} = 0$. Consider now $f \in V_1$, $g \in V_2$. Then we have

$$\langle f, g \rangle = \langle P_{V_1} f, P_{V_2} g \rangle = \langle f, P_{V_1} P_{V_2} g \rangle = 0.$$

Given that f and g were generic this implies that $V_1 \perp V_2$.

□

3.2 Exercise 2 - Derivative of inner product (complement to the class)

Let $\phi(t)$ and $\psi(t)$ differentiable functions on the Hilbert space $\mathcal{H}$, meaning that the limit

$$\frac{d\phi}{dt}(t) := \lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} \tag{28}$$

exists in the norm topology of $\mathcal{H}$ for each $t \in \mathbb{R}$, and similarly for $\psi(t)$.

Prove that

$$\frac{d}{dt}\langle \phi(t), \psi(t) \rangle = \langle \frac{d\phi}{dt}(t), \psi(t) \rangle + \langle \phi(t), \frac{d\psi}{dt}(t) \rangle \tag{29}$$

Proof. First notice that (28) means that

$$\lim_{h \to 0} \left\| \frac{d\phi}{dt}(t) - \frac{\phi(t+h) - \phi(t)}{h} \right\| = 0.$$

In particular this implies that

$$\lim_{h \to 0} \|\phi(t+h) - \phi(t)\| \leqslant \lim_{h \to 0} |h| \left(\left\| \frac{d\phi}{dt}(t) - \frac{\phi(t+h) - \phi(t)}{h} \right\| + \left\| \frac{d\phi}{dt}(t) \right\| \right) = 0,$$

and therefore $\phi(t)$ is also continuous in the norm topology of $\mathcal{H}$, and similarly for $\psi(t)$.

Consider now (29); we get

$$\frac{d}{dt}\langle \phi(t), \psi(t) \rangle = \lim_{h \to 0} \frac{\langle \phi(t+h), \psi(t+h) \rangle - \langle \phi(t), \psi(t) \rangle}{h}.$$

The term inside the limit can be decomposed as follows:

$$\begin{aligned}
&\frac{1}{h}\left(\langle\phi(t+h),\psi(t+h)\rangle-\langle\phi(t),\psi(t)\rangle\right)=\\
&\qquad=\frac{1}{h}\left(\langle\phi(t+h)-\phi(t),\psi(t+h)\rangle+\langle\phi(t),\psi(t+h)-\psi(t)\rangle\right)\\
&\qquad=\langle\frac{\phi(t+h)-\phi(t)}{h},\psi(t+h)-\psi(t)\rangle+\langle\frac{\phi(t+h)-\phi(t)}{h},\psi(t)\rangle\\
&\qquad\quad+\langle\phi(t),\frac{\psi(t+h)-\psi(t)}{h}\rangle.
\end{aligned}$$

We now study the limit of these three terms. The first one can be bound completely, so we can apply Cauchy-Schwarz to get

$$\begin{aligned}
\lim_{h\to 0}\left|\langle\frac{\phi(t+h)-\phi(t)}{h},\psi(t+h)-\psi(t)\rangle\right| &\leqslant \lim_{h\to 0}\left\|\frac{\phi(t+h)-\phi(t)}{h}\right\|\|\psi(t+h)-\psi(t)\|\\
&=\lim_{h\to 0}\left\|\frac{d\phi}{dt}(t)\right\|\|\psi(t+h)-\psi(t)\|=0.
\end{aligned}$$

For the second term one can proceed as follows. Using the fact that $\phi(t)$ is differentiable and applying Cauchy-Schwarz again we get

$$\begin{aligned}
&\lim_{h\to 0}\left|\langle\frac{\phi(t+h)-\phi(t)}{h},\psi(t)\rangle-\langle\frac{d\phi}{dt}(t),\psi(t)\rangle\right|\leqslant\\
&\qquad\leqslant\lim_{h\to 0}\left\|\frac{\phi(t+h)-\phi(t)}{h}-\frac{d\phi}{dt}(t)\right\|\|\psi(t)\|=0.
\end{aligned}$$

Proceeding similarly for the third term we get the result.

□

3.3 Exercise 3 - $\frac{1}{i\hbar}[A,B]$ is self-adjoint

Let $\mathcal{H}$ be a Hilbert space. Consider A and B bounded self-adjoint operators on $\mathcal{H}$. Prove that $\frac{1}{i\hbar}[A,B]$ is self adjoint.

Proof. Recall that in the last exercise sheet we proved that $(AB)^*=B^*A^*$ and that $(\alpha A)^*=\overline{\alpha}A^*$ for any A, B bounded operators on $\mathcal{H}$ and for any $\alpha\in\mathbb{C}$. We therefore get

$$\begin{aligned}
\left(\frac{1}{i\hbar}[A,B]\right)^* &= -\frac{1}{i\hbar}[A,B]^*=-\frac{1}{i\hbar}(AB-BA)^*=-\frac{1}{i\hbar}(B^*A^*-A^*B^*)\\
&=-\frac{1}{i\hbar}(BA-AB)=-\frac{1}{i\hbar}[B,A]=\frac{1}{i\hbar}[A,B].
\end{aligned}$$

□

3.4 Exercise 4 - Properties of the commutator

Consider a vector space V over $\mathbb{C}$, $A,\ B,\ C$ linear bounded operators on V and $\alpha \in \mathbb{C}$.

a Prove that $[A, B + \alpha C] = [A, B] + \alpha\,[A, C]$.

b Prove that $[B, A] = -\,[A, B]$.

c Prove that $[A, BC] = [A, B]\,C + B\,[A, C]$.

d Prove that $[A, [B, C]] = [[A, B]\,, C] + [B, [A, C]]$.

Proof. To prove **a** notice that

$$\begin{aligned}[A, B + \alpha C] &= A\,(B + \alpha C) - (B + \alpha C)\,A = AB - BA + \alpha AC - \alpha CA\\ &= [A, B] + \alpha\,[A, C]\,.\end{aligned}$$

To prove **b** one can see that

$$[B, A] = BA - AB = -\,(AB - BA) = -\,[A, B]\,.$$

To prove **c** we look at the right side to get

$$\begin{aligned}[A, B]\,C + B\,[A, C] &= (AB - BA)\,C + B\,(AC - CA)\\ &= ABC - BAC + BAC - BCA = [A, BC]\,.\end{aligned}$$

To prove **d** we notice that

$$\begin{aligned}&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] =\\ &\qquad = A\,(BC - CB) - (BC - CB)\,A\\ &\qquad + B\,(CA - AC) - (CA - AC)\,B\\ &\qquad + C\,(AB - BA) - (AB - BA)\,C = 0.\end{aligned}$$

This implies in particular

$$[A, [B, C]] = -\,[B, [C, A]] - [C, [A, B]] = [[A, B]\,, C] + [B, [A, C]]\,.$$

□

4 Exercise Sheet 4

4.1 Exercise 1 - Two bounded operator cannot commute in a nontrivial manner

Let $\mathcal{H}$ be an Hilbert space. Let A and B linear operators on $\mathcal{H}$ such that there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that

$$[A, B] = \alpha \,\mathrm{id}\,. \tag{30}$$

Prove that A and B cannot be both bounded.

Hint: Assume both bounded; consider $\|[A, B^n]\|$ and find an absurd.

Proof. Assume that both A and B are bounded operators. Consider for any $n \in \mathbb{N}$ the commutator between A and B^n. We have

$$[A, B^n] = \left[A, BB^{n-1}\right] = [A, B]\, B^{n-1} + B\left[A, B^{n-1}\right] = \alpha B^{n-1} + B\left[A, B^{n-1}\right].$$

We can then prove by induction that $[A, B^n] = n\alpha B^{n-1}$; indeed if $n = 1$ the statement is trivially true, and if we assume the statement to be true for $n-1$ we get

$$[A, B^n] = \alpha B^{n-1} + B\left[A, B^{n-1}\right] = \alpha B^{n-1} + B\left((n-1)\,\alpha B^{n-2}\right) = n\alpha B^{n-1}.$$

Consider now the norm of the commutator; we get

$$\begin{aligned}\|[A, B^n]\| &= \|AB^n - B^n A\| \leqslant 2\,\|A\|\,\|B^n\| \\ &\leqslant \|A\|\,\|B\|^n\,.\end{aligned}$$

Given that $[A, B] \neq 0$ we can deduce that $\|A\| \neq 0$. We then get that

$$\|B^n\| \geqslant \frac{\alpha}{2\,\|A\|} n \left\|B^{n-1}\right\| \geqslant \ldots \geqslant \left(\frac{\alpha}{2\,\|A\|}\right)^{n-1} n!\,\|B\| > 0$$

and from this we deduce that for any $n \in \mathbb{N}$ we have $B^n \neq 0$. We then get

$$n\,|\alpha|\left\|B^{n-1}\right\| \leqslant 2\,\|A\|\,\|B^n\| \leqslant 2\,\|A\|\,\|B\|\left\|B^{n-1}\right\| \quad\Longrightarrow\quad n\,|\alpha| \leqslant \|A\|\,\|B\|\,.$$

Given that the last inequality holds for any n this gives us a contradiction.

□

4.2 Exercise 2 - Fourier transform of the complex gaussian

a Prove that for any $\alpha \in \mathbb{C}$ such that $\mathrm{Re}\,(\alpha) > 0$,

$$\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx\right)^2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2\alpha}} dxdy \tag{31}$$

$$= 2\pi\alpha, \tag{32}$$

where the integral over $\mathbb{R}^2$ can be evaluated using polar coordinates. Deduce that

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx = \sqrt{2\pi\alpha}, \tag{33}$$

where the square root is the one with positive real part.

b For all $B \geqslant A > 0$ and $\alpha \in \mathbb{C}\setminus\{0\}$ we have

$$\int_A^B e^{-\frac{x^2}{2\alpha}} dx = -\left.\frac{\alpha}{x} e^{-\frac{x^2}{2\alpha}}\right|_A^B - \int_A^B \frac{\alpha}{x^2} e^{-\frac{x^2}{2\alpha}} dx. \tag{34}$$

Using this, prove that the integral in (33) is convergent for all nonzero α with $\mathrm{Re}(\alpha) \geqslant 0$, provided the integral is interpreted as a principle value when not absolutely convergent, where the principal value is defined as

$$\mathrm{PV}\int_{\mathbb{R}} f(x)\, dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{35}$$

c Prove that the result of **a** is also valid for nonzero values of α with $\mathrm{Re}(\alpha) = 0$, at least in the principal value.

Hint: Given $\eta \neq 0$, show that the principal value from A to $+\infty$ of $\exp\left[-\frac{x^2}{2(\gamma+i\eta)}\right]$ *is small for large A, uniformly in $\gamma \in [0,1]$.*

d Prove that

$$\frac{1}{2\pi}\mathrm{PV}\int_{\mathbb{R}} e^{ikx} e^{-i\frac{\hbar t}{2m}k^2} dk = \sqrt{\frac{m}{2\pi i\hbar t}} e^{i\frac{m}{2\hbar t}x^2}, \tag{36}$$

where the square root is the one with real positive part.

Proof. We start by proving point **a**. Using polar coordinates we get

$$\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx\right)^2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2\alpha}} dxdy = 2\pi\int_0^{+\infty} e^{-\frac{\rho^2}{2\alpha}} \rho d\rho = -\left. 2\pi\alpha e^{-\frac{\rho^2}{2\alpha}}\right|_0^{+\infty} = 2\pi\alpha.$$

To recover the integral we want is enough to apply the square root, and given that for real values of α the integral we get is positive, we choose the positive determination of the square root to get

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx = \sqrt{2\pi\alpha}.$$

To prove **b** we first use (34) to estimate the principal value. Fix $A > 0$; then we get

$$\begin{aligned}
\mathrm{PV}\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx &= \lim_{R\to+\infty}\int_{-R}^{R} e^{-\frac{x^2}{2\alpha}} dx = \int_{-A}^{A} e^{-\frac{x^2}{2\alpha}} dx + 2\lim_{R\to+\infty}\int_A^R e^{-\frac{x^2}{2\alpha}} dx \\
&= \int_{-A}^{A} e^{-\frac{x^2}{2\alpha}} dx + 2\lim_{R\to+\infty}\left[-\left.\frac{\alpha}{x}e^{-\frac{x^2}{2\alpha}}\right|_A^R - \int_A^R \frac{\alpha}{x^2} e^{-\frac{x^2}{2\alpha}} dx\right] \\
&= \int_{-A}^{A} e^{-\frac{x^2}{2\alpha}} dx + 2\lim_{R\to+\infty}\left[-\frac{\alpha}{R}e^{-\frac{R^2}{2\alpha}} + \frac{\alpha}{A}e^{-\frac{A^2}{2\alpha}} - \int_A^R \frac{\alpha}{x^2} e^{-\frac{x^2}{2\alpha}} dx\right].
\end{aligned}$$

Now using that $\mathrm{Re}\,(\alpha) \geqslant 0$ we have that

$$\left|e^{-\frac{R^2}{2\alpha}}\right| \leqslant e^{-\mathrm{Re}\left(\frac{R^2}{2\alpha}\right)} \leqslant 1, \qquad \left|\frac{1}{x^2}e^{-\frac{R^2}{2\alpha}}\right| \in L^1\left(A, +\infty\right)$$

and applying this to the limit we get that

$$\mathrm{PV}\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}}\,dx = \int_{-A}^{A} e^{-\frac{x^2}{2\alpha}}\,dx + \frac{2\alpha}{A}e^{-\frac{A^2}{2\alpha}} - \int_{A}^{+\infty}\frac{2\alpha}{x^2}e^{-\frac{x^2}{2\alpha}}\,dx,$$

and therefore the integral is convergent for any α with $\mathrm{Re}\,(\alpha) \geqslant 0$.

To prove **c** is enough to consider $\alpha = i\eta$ with $\eta \in \mathbb{R}\backslash\{0\}$. In this case we get

$$\begin{aligned}\mathrm{PV}\int_{\mathbb{R}} e^{i\frac{x^2}{2\eta}}\,dx &= \lim_{R\to+\infty}\int_{-R}^{R} e^{i\frac{x^2}{2\eta}}\,dx = \lim_{R\to+\infty}\int_{-R}^{R}\lim_{\gamma\to0^+} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\\ &= \lim_{R\to+\infty}\lim_{\gamma\to0^+}\int_{-R}^{R} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx,\end{aligned}$$

where in the last equality we could bring the limit outside of the integral because the integrand is uniformly bounded in modulus by 1 which is integrable in $[-R, R]$. From the formula above and from the fact that now $\mathrm{Re}\,(\gamma + i\eta) > 0$ we now have that

$$\int_{-R}^{R} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx = \sqrt{2\pi\left(\gamma+i\eta\right)} - 2\int_{R}^{+\infty} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx.$$

Moreover we can assume that $\gamma \in [0, 1]$ and use **b** to get that

$$\begin{aligned}\left|\int_{R}^{+\infty} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\right| &= \lim_{L\to+\infty}\left|\int_{R}^{L} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\right|\\ &= \lim_{L\to+\infty}\left|-\frac{\gamma+i\eta}{L}e^{-\frac{L^2}{2(\gamma+i\eta)}} + \frac{\gamma+i\eta}{R}e^{-\frac{R^2}{2(\gamma+i\eta)}} - \int_{R}^{L}\frac{\gamma+i\eta}{x^2}e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\right|\\ &= \left|\frac{\gamma+i\eta}{R}e^{-\frac{R^2}{2(\gamma+i\eta)}} - \int_{R}^{+\infty}\frac{\gamma+i\eta}{x^2}e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\right|\\ &\leqslant \frac{4}{R}\left|\gamma+i\eta\right| e^{-\frac{R^2}{2}\mathrm{Re}\left(\frac{1}{\gamma+i\eta}\right)} \leqslant \frac{4\sqrt{1+\eta^2}}{R},\end{aligned}$$

and therefore, passing to the limit we get

$$\left|\lim_{R\to+\infty}\lim_{\gamma\to0^+}\int_{R}^{+\infty} e^{-\frac{x^2}{2(\gamma+i\eta)}}\,dx\right| \leqslant \lim_{R\to+\infty}\lim_{\gamma\to0^+}\frac{4\sqrt{1+\eta^2}}{R} = 0.$$

As a consequence we get that

$$\mathrm{PV}\int_{\mathbb{R}} e^{i\frac{x^2}{2\eta}}\,dx = \lim_{R\to+\infty}\lim_{\gamma\to0^+}\sqrt{2\pi\left(\gamma+i\eta\right)} = \sqrt{2\pi i\eta},$$

which concludes the proof of **c**.

To prove **d** we first notice that

$$\frac{\hbar t}{2m}k^2 - kx = \frac{\hbar t}{2m}\left(k - \frac{mx}{\hbar t}\right)^2 - \frac{mx^2}{2\hbar t}.$$

Using **c** we then get

$$\frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} e^{ikx}e^{-i\frac{\hbar t}{2m}k^2}dk = \frac{e^{i\frac{m}{2\hbar t}x^2}}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} e^{-i\frac{\hbar t}{2m}k^2}dk$$
$$= \sqrt{\frac{m}{2\pi i\hbar t}}e^{i\frac{m}{2\hbar t}x^2},$$

which concludes our proof.

□

4.3 Exercise 3 - Counterexample for the closed graph theorem

Consider a separable Hilbert space $\mathcal{H}$ and a complete orthonormal system for it $\{\varphi_n\}_{n\in\mathbb{N}}$. Assume that φ_∞ cannot be written as a finite linear combination of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$. Let D denote the dense linear subspace of $\mathcal{H}$ consisting of all finite linear combinations of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$ and of φ_∞. On D define the operator $T : D \to \mathcal{H}$ defined as

$$T\left(\alpha_\infty\varphi_\infty + \sum_{n\in\mathbb{N}}\alpha_n\varphi_n\right) := \alpha_\infty\varphi_\infty. \tag{37}$$

Prove that T is not bounded.

Hint: Use the closed graph theorem.

Proof. Suppose that T is bounded. Given that D is dense in $\mathcal{H}$, we can define $\widetilde{T}$ an extension of T to $\mathcal{H}$. Consider now the graph of $\widetilde{T}$; given that $\{\varphi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system there exists a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{N\to+\infty}\sum_{n=0}^{N}\beta_n\varphi_n = \varphi_\infty.$$

Recall the definition of $G\left(\widetilde{T}\right) := \left\{\left(\psi, \widetilde{T}\psi\right) \mid \psi \in \mathcal{H}\right\} \subseteq \mathcal{H}\times\mathcal{H}$. Given that we have that

$$T\left(\sum_{n=0}^{N}\beta_n\varphi_n\right) = 0,$$

we get that $\left(\sum_{n=0}^{N}\beta_n\varphi_n, 0\right) \in G\left(\widetilde{T}\right)$, and as a consequence $(\varphi_\infty, 0) \in \overline{G\left(\widetilde{T}\right)}$. On the other hand, by definition of T we get that $T\varphi_\infty = \varphi_\infty$, and therefore that $(\varphi_\infty, 0) \notin G\left(\widetilde{T}\right)$. For this reason we get that $G\left(\widetilde{T}\right) \neq \overline{G\left(\widetilde{T}\right)}$. On the other hand $\widetilde{T}$ is trivially linear on $\mathcal{H}$, so we can apply the closed graph theorem to imply that T cannot be bounded.

□

4.4 Exercise 4 - Free Schrödinger equation preserves the domain

Recall the definition of $H^2(\mathbb{R})$ as

$$H^2(\mathbb{R}) := \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}$$

Recall that in class we defined the map that to any initial datum $\psi_0 \in L^2(\mathbb{R})$ would associate $\psi_t := \widetilde{U}_0(t)\psi_0$, defined via the Hamiltonian $H_0 := -\frac{\partial^2}{\partial x^2}$ with domain $\mathcal{D}(H_0) = H^2(\mathbb{R})$. Indeed if $U_0(t)\psi_0$ is defined for any $\psi_0 \in \mathcal{S}(\mathbb{R})$ as the unique solution to

$$\begin{cases} i\hbar\partial_t(U_0(t)\psi_0) = H_0U_0(t)\psi_0 \\ U_0(t)\psi_0|_{t=0} = \psi_0, \end{cases} \tag{38}$$

then $\widetilde{U}_0(t)$ is defined by density on the whole space $L^2(\mathbb{R})$, and coincides with $U_0(t)$ on $\mathcal{S}(\mathbb{R})$.

Prove that if $\psi_0 \in D(H_0)$ then $\psi_t \in \mathcal{D}(H_0)$.

Proof. We saw in class that $\widetilde{U}_0(t)$ has an explicit form; indeed for any $\psi_0 \in L^2(\mathbb{R})$ we get thet

$$\mathcal{F}\left(\widetilde{U}_0(t)\psi_0\right)(k) = e^{-i\frac{\hbar t}{2m}k^2}\widehat{\psi}_0(x),$$

where $\mathcal{F}$ indicates the Fourier transform operator

Now, if $\psi_0 \in H^2(\mathbb{R})$, we get by definition that $k^2\psi_0 \in L^2(\mathbb{R})$. As a consequence we also get $k^2\mathcal{F}\left(\widetilde{U}_0(t)\psi_0\right) \in L^2(\mathbb{R})$, and therefore $\widetilde{U}_0(t)\psi_0 \in H^2(\mathbb{R})$.

□

5 Exercise Sheet 5

5.1 Exercise 1 - Well-posedness of standard deviation

Let ψ be a unit vector in $L^2(\mathbb{R})$ such that $x\psi,\ x^2\psi \in L^2(\mathbb{R})$. Prove that

$$\langle X^2\rangle_\psi \geqslant \left(\langle X\rangle_\psi\right)^2, \tag{39}$$

where as we defined in class, X is the operator given by the multliplication by x and

$$\langle A\rangle_\psi := \langle \psi, A\psi\rangle. \tag{40}$$

Hint: Use Jensen inequality.

Proof. Recall that Jensen inequality states that if μ is a probability measure on a measurable space Ω, f is a real valued function and Ξ is a convex function from $\mathbb{R}$ to itself, then we have

$$\Xi\left(\int_\Omega f(x)\,d\mu(x)\right) \leqslant \int_\Omega \Xi\circ f(x)\,d\mu(x).$$

Consider now the space $\Omega = \mathbb{R}$. The measure $|\psi(x)|^2\,dx$ is a probability measure because ψ has L^2-norm equal to 1. Now, if we consider $f(x) = x$ and $\Xi(t) = t^2$ in Jensen inequality we get

$$\left(\langle X\rangle_\psi\right)^2 = \left(\int_\mathbb{R} x\,|\psi(x)|^2\,dx\right)^2 \leqslant \int_\mathbb{R} x^2\,|\psi(x)|^2\,dx = \langle X^2\rangle_\psi.$$

□

5.2 Exercise 2 - Operator norm of multiplication by a sequence

Let $\alpha := \{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers. Consider the Hilbert space of the square integrable functions $\mathfrak{h} := l^2(\mathbb{Z})$. Consider the operator that to the sequence $x := \{x_n\}_{n\in\mathbb{Z}}$ associate the sequence $M_\alpha x = \{\alpha_n x_n\}_{n\in\mathbb{Z}}$.

Suppose that $\|\alpha\|_\infty := \sup_{n\in\mathbb{Z}} |\alpha_n| < +\infty$. Prove that M_a is a well defined linear bounded operator from $\mathfrak{h}$ to itself and prove that $\|M_\alpha\| = \|\alpha\|_\infty$.

Proof. First notice that for any element of the sequence $M_\alpha x$ we get $|\alpha_n x_n| \leqslant \|\alpha\|_\infty |x_n|$. As a consequence we get

$$\|M_\alpha x\|_\mathfrak{h} = \left(\sum_{n\in\mathbb{Z}} |\alpha_n x_n|^2\right)^{\frac{1}{2}} \leqslant \|\alpha\|_\infty \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{\frac{1}{2}} = \|\alpha\|_\infty \|x\|_\mathfrak{h}.$$

Therefore M_α is well defined from $\mathfrak{h}$ to itself and it is trivially linear. From the previous inequality we also get that $\|M_\alpha\| \leqslant \|\alpha\|_\infty$.

To prove the equality, first define for any $j \in \mathbb{Z}$ the element $e_j := \{\delta_{j,n}\}_{n\in\mathbb{Z}} \in \mathfrak{h}$. We get that $\|e_j\|_{\mathfrak{h}} = 1$ and that $M_\alpha e_j = \alpha_j e_j$. Now, by definition of sup there is a sequence $\{n_j\}_{j\in\mathbb{N}}$ such that $|\alpha_{n_j}| \to \|\alpha\|_\infty$ as $j \to +\infty$, and we then get

$$\|\alpha\|_\infty = \lim_{j\to+\infty} |\alpha_{n_j}| = \lim_{j\to+\infty} \|M_\alpha e_{n_j}\|_{\mathfrak{h}} \leqslant \lim_{j\to+\infty} \|M_\alpha\| \|e_{n_j}\|_{\mathfrak{h}} = \|M_\alpha\|,$$

concluding the proof.

□

5.3 Exercise 3 - No solutions for too low energy in the potential well (complement to the class)

Consider the Hilbert space $\mathfrak{h} := L^2(\mathbb{R})$. And the operator H define

$$\mathcal{D}(H) := H^2(\mathbb{R}) = \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}$$
$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(X),$$

where the operator $(V(X)\psi)(x) = V(x)\psi(x)$, with

$$V(x) := \begin{cases} -C & \text{if } |x| \leqslant A, \\ 0 & \text{if } |x| > A, \end{cases} \tag{41}$$

and with A and C positive constants. Consider $E \in (-\infty, -C]$ and prove that there is no nonzero $\psi_E \in \mathcal{D}(H)$ such that

$$H\psi_E = E\psi_E. \tag{42}$$

Proof. Suppose there exists E such in the text of the exercise. Given that $\psi_E \neq 0$ we can assume that $\|\psi_E\|_{\mathfrak{h}} = 1$ As a consequence we get

$$E = \langle \psi_E, E\psi_E\rangle = \langle \psi_E, H\psi_E\rangle = \langle \psi_E, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E\rangle + \langle \psi_E, V(X)\psi_E\rangle.$$

Given that $\psi_E \in \mathcal{D}(H)$ we can integrate by part the first term and obtain

$$\langle \psi_E, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E\rangle = \frac{\hbar^2}{2m}\langle \frac{\partial}{\partial x}\psi_E, \frac{\partial}{\partial x}\psi_E\rangle = \frac{\hbar^2}{2m}\left\|\frac{\partial}{\partial x}\psi_E\right\| \geqslant 0.$$

On the other hand we have

$$\langle \psi_E, V(X)\psi_E\rangle \geqslant -|\langle \psi_E, V(X)\psi_E\rangle| \geqslant -\|V\|_\infty \|\psi_E\|_{\mathfrak{h}}^2 = -C.$$

Given that $E \in (-\infty, -C]$ we get

$$-C \geqslant E \geqslant \langle \psi_E, -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E\rangle + \langle \psi_E, V(X)\psi_E\rangle \geqslant \langle \psi_E, V(X)\psi_E\rangle \geqslant -C,$$

and therefore $E = -C$.

Now as we saw in class the function ψ_E needs to satisfy the following equation

$$\begin{cases} -\dfrac{\hbar^2}{2m}\psi_E'' = -C\psi_E & \text{if } |x| \leqslant A, \\ -\dfrac{\hbar^2}{2m}\psi_E'' = (C+E)\,\psi_E = 0 & \text{if } |x| > A, \\ \lim\limits_{x\to\pm A^-} \psi_E(x) = \lim\limits_{x\to\pm A^+} \psi_E(x), \\ \lim\limits_{x\to\pm A^-} \psi_E'(x) = \lim\limits_{x\to\pm A^+} \psi_E'(x). \end{cases}$$

Suppose now $x \in (-\infty, -A)$. Then we get $\psi_E = c_0 + c_1 x$. Given that $\psi_E \in \mathfrak{h} = L^2(\mathbb{R})$, we then get that $c_0 = c_1 = 0$. Proceeding similarly for $x \in (A, +\infty)$ we get that $\psi_E(x) = 0$ for any $|x| > A$.

So ψ_E solves

$$\begin{cases} \psi_E'' = \dfrac{2mC}{\hbar^2}\psi_E & \text{if } |x| \leqslant A, \\ \psi_E(\pm A) = \psi_E'(\pm A) = 0. \end{cases}$$

Now the solution to the differential equation is $\psi_E(x) = c_+ e^{\left(\sqrt{2mC}/\hbar\right)x} + c_- e^{-\left(\sqrt{2mC}/\hbar\right)x}$. From the fact that $\psi_E(-A) = \psi_E(A)$ we get

$$(c_+ - c_-)\sinh\left(\frac{\sqrt{2mC}}{\hbar}A\right) = 0,$$

which in particular implies $c_+ = c_-$. As a consequence we get

$$\psi_E(x) = 2c_+\cosh\left(\frac{\sqrt{2mC}}{\hbar}x\right).$$

Using the fact that $\psi_E(A) = 0$ we get $c_+ = 0$, implying that the unique eigenfunction corresponding to E is the zero vector, which is absurd and concludes our proof.

□

5.4 Exercise 4 - Odd solutions to the potential well (complement to the class)

Let $\mathfrak{h}$, H and $\mathcal{D}(H)$ as in Exercise 3. In class we saw that for any $E \in (-C, 0)$ there is always at least one nonzero even solution ψ_E to the problem $H\psi_E = E\psi_E$.

Prove that if $A\sqrt{2mC}\hbar \leqslant \frac{\pi}{2}$ there are no nonzero odd solutions, and for larger values of C there is always at least one.

Proof. Proceeding as in class it is easy to see that any odd solution ψ_E to $H\psi_E = E\psi_E$ is such that

$$\psi_E(x) = \begin{cases} ce^{-\frac{\sqrt{2m|E|}}{\hbar}(x-A)} & \text{if } x > A, \\ -ce^{\frac{\sqrt{2m|E|}}{\hbar}(x+A)} & \text{if } x > A. \end{cases}$$

This explicit form of the solution outside the ball $|x| \leqslant A$ gives us boundary conditions for the problem that the solution needs to solve inside the ball:

$$\begin{cases} -\dfrac{\hbar^2}{2m}\psi_E'' = (C+E)\,\psi_E, \\ \psi_E\,(\pm A) = \pm c, \\ \psi_E'\,(\pm A) = -\dfrac{\sqrt{2m\,|E|}}{\hbar}c. \end{cases}$$

Out of convenience, we define, similarly as in class, the constants $\kappa := (2mC)/\hbar^2$ and $\varepsilon := -(2mE)/\hbar^2$. We then have that $E \in (-C, 0)$ if and only if $\varepsilon \in (0, \kappa)$.

We are then looking for the odd solution to the problem

$$\begin{cases} -\psi_E'' = (\kappa-\varepsilon)\,\psi_E, \\ \psi_E\,(\pm A) = \pm c, \\ \psi_E'\,(\pm A) = -\sqrt{\varepsilon}c. \end{cases}$$

A generic solution for this problem is of the form $\psi_E\,(x) = \alpha \sin\left(\sqrt{\kappa-\varepsilon}x\right) + \beta\cos\left(\sqrt{\kappa-\varepsilon}x\right)$, with α and β to be determined. Given that our function is odd, we have that $\beta = 0$. The boudnary conditions then gives us the following relations:

$$\begin{cases} \alpha \sin\left(\sqrt{\kappa-\varepsilon}A\right) = c, \\ \alpha\sqrt{\kappa-\varepsilon}\cos\left(\sqrt{\kappa-\varepsilon}A\right) = -\sqrt{\varepsilon}c. \end{cases}$$

If $c = 0$, the first equation tells us that if we do not want the trivial solution, $\sqrt{\kappa-\varepsilon}A = \eta\pi$, with $\eta \in \mathbb{Z}$. This implies that $\cos\left(\sqrt{\kappa-\varepsilon}A\right) = \pm 1$, and applying this to the second equation we would deduce that $\kappa = \varepsilon$, which is not possible. So $c \neq 0$ if and only if $\alpha \neq 0$. Suppose then $c \neq 0$ (and therefore $\alpha \neq 0$). Dividing the second equation by the first one we then get the following matching condition

$$\sqrt{\kappa-\varepsilon}\cot\left(\sqrt{\kappa-\varepsilon}A\right) = -\sqrt{\varepsilon}.$$

Now, if $\sqrt{\kappa}A \leqslant \frac{\pi}{2}$ we get that $\sqrt{\kappa-\varepsilon}A \in \left(0, \frac{\pi}{2}\right)$, and as a consequence the term on the left of the matching condition is strictly positive. On the other hand the term on the right is strictly negative, therefore the matching condition cannot be satisfied and there is no odd solution to the problem.

Consider now $\sqrt{\kappa}A > \frac{\pi}{2}$; define the interval $I := \left(\max\left\{0, k-\pi^2/A^2\right\}, k-\pi^2/4A^2\right)$ and the following mapping:

$$\begin{array}{rccl} \xi: & I & \to & \mathbb{R} \\ & \varepsilon & \mapsto & \sqrt{\varepsilon} + \sqrt{\kappa-\varepsilon}\cot\left(\sqrt{\kappa-\varepsilon}A\right). \end{array}$$

If $\max\left\{0, k-\pi^2/A^2\right\} = 0$ then we have that $\sqrt{\kappa}A \leqslant \pi$ and $\cot\left(\sqrt{\kappa-\varepsilon}A\right) \in (-\infty, 0)$; in particular

$$\xi\,(I) = \left(-\sqrt{\kappa}\left|\cot\left(\sqrt{\kappa}A\right)\right|, \sqrt{\kappa-\frac{\pi^2}{A^2}}\right).$$

If $\max\left\{0, k - \pi^2/A^2\right\} = k - \pi^2/A^2$ then we have that

$$\xi\left(I\right) = \left(-\infty, \sqrt{\kappa - \frac{\pi^2}{A^2}}\right).$$

In both cases $0 \in \xi\left(I\right)$ and we have that there is a solution to the matching conditions, which implies the existence of a nontrivial odd solution. □

6 Exercise Sheet 6

6.1 Exercise 1 - A preserves a space, A^* preserves the orthogonal

Let V be a closed subspace of $\mathcal{H}$ Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ such that $A(V) \subseteq V$. Prove that $A^*\left(V^\perp\right) \subseteq V^\perp$.

Proof. Consider $\psi \in V^\perp$ and let $\varphi \in V$. We then get

$$\langle \varphi, A^*\psi \rangle = \langle A\varphi, \psi \rangle = 0,$$

because $A\varphi \in V$ and $\psi \in V^\perp$. Given that φ was generic, we get that $A^*\psi \in V^\perp$.

□

6.2 Exercise 2 - Inverse of the adjoint of an invertible

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ with linear bounded inverse A^{-1}. Prove that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$. Deduce that A^* is invertible and that $(A^*)^{-1} = \left(A^{-1}\right)^*$.

Proof. Given that A is invertible we get that both A^* and $\left(A^{-1}\right)^*$ are well defined linear bounded operators. Recall that we proved before (see Exercise Sheet number 2) that $(AB)^* = B^*A^*$. We then get $\mathrm{id} = \mathrm{id}^* = \left(AA^{-1}\right)^* = \left(A^{-1}\right)^* A^*$. In a similar way, we also get $\mathrm{id} = \mathrm{id}^* = \left(A^{-1}A\right)^* = A^*\left(A^{-1}\right)^*$.

Now, given that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$ then A^* is invertible and $(A^*)^{-1} = \left(A^{-1}\right)^*$.

□

6.3 Exercise 3 - Creation, annihilation and number

Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$.

a Define the operator A as

$$(A\alpha)_n = \alpha_{n+1} \qquad \forall n \in \mathbb{N}, \tag{43}$$

for any $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{H}$.

Prove that A is a well defined linear bounded operator, find its norm and its spectrum.

b Consider A^* the adjoint of A. Show its explicit action and find its norm and its spectrum.

c Define $B := A^*A$. Prove that B is a self-adjoint operator, show its explicit action and find its norm and its spectrum.

Hint: Recall that if T is a linear bounded operator, the spectrum $\sigma(T)$ is a closed set, $\rho(T) \equiv \mathbb{C} \setminus \sigma(T)$ the resolvent of T is defined as

$$\rho(T) := \left\{ \lambda \in \mathbb{C} | \ (T - \lambda \, \mathrm{id})^{-1} \textit{ is a well-defined, linear, bounded operator} \right\}, \tag{44}$$

and that $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$, where $B_R(0) := \{\alpha \in \mathcal{H} | \ \|\alpha\|_2 < R\}$.

Proof. To prove **a**, first consider $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$, $\beta = \{\beta_n\}_{n \in \mathbb{N}} \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. We get

$$(A(\alpha + \lambda\beta))_n = (\alpha + \lambda\beta)_{n+1} = \alpha_{n+1} + \lambda\beta_{n+1} = (A\alpha)_n + \lambda (A\beta)_n,$$

and therefore A is linear. To prove that is bounded consider $\alpha \in \mathcal{H}$; we get

$$\|A\alpha\|_2^2 = \sum_{n \geqslant 0} |(A\alpha)_n|^2 = \sum_{n \geqslant 1} |\alpha_n|^2 \leqslant \|\alpha\|_2^2,$$

therefore A is well defined from $\mathcal{H}$ to itself and $\|A\| \leqslant 1$. Let now $e^j = \{\delta_{j,n}\}_{n \in \mathbb{N}}$; on the one hand $\|e^j\|_2 = 1$, on the other we also get that for any $j > 0$ we get $\|Ae^j\|_2 = 1$, therefore $\|A\| = 1$.

Given that $\|A\| = 1$ we get that $\sigma(A) \subseteq \overline{B_1(0)}$. Consider now $\lambda \in B_1(0)$. If we look for a solution of $A\alpha = \lambda\alpha$, we get that such α needs to satisfy

$$\alpha_{n+1} = \lambda\alpha_n.$$

It is easy to see that $\alpha_n := \lambda^n \alpha_0$ satisfies the equation, and given that

$$\|\alpha\|_2^2 = \sum_{n \geqslant 0} |\lambda|^{2n} |\alpha_0|^2 = \frac{|\alpha_0|^2}{1 - |\lambda|^2}$$

we also get that $\alpha \in \mathcal{H}$. This implies that α is an eigenvector for A and as a consequence $B_1(0) \subseteq \sigma(A)$. Given that the spectrum is always a closed set we get $\overline{B_1(0)} \subseteq \overline{\sigma(A)} = \sigma(A) \subseteq \overline{B_1(0)}$, and hence $\sigma(A) = \overline{B_1(0)}$.

To prove **b**, let $\alpha, \ \gamma \in \mathcal{H}$. We get

$$\sum_{n \geqslant 0} \overline{\gamma_n} (A^*\alpha)_n = \langle \gamma, A^*\alpha \rangle = \langle A\gamma, \alpha \rangle = \sum_{n \geqslant 0} \overline{(A\gamma)_n} \alpha_n = \sum_{n \geqslant 0} \overline{\gamma_{n+1}} \alpha_n = \sum_{n \geqslant 1} \overline{\gamma_n} \alpha_{n-1}.$$

Given that α and γ were arbitrary we get that

$$(A^*\alpha)_n := (1 - \delta_{n,0}) \alpha_{n-1} \equiv \begin{cases} \alpha_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

From the definition we easily get that $\|A^*\alpha\|_2 = \|\alpha\|_2$, and therefore $\|A^*\| = 1$.

If we now turn to the spectrum, we get that given that $\|A^*\| = 1$, then $\sigma(A^*) \subseteq \overline{B_1(0)}$. Consider now $\lambda \in B_1(0)$ and let $\gamma \in \mathcal{H}$. We look for α so that $(A^* - \lambda \, \mathrm{id}) \alpha = \gamma$. Then we have

$$\begin{aligned} \alpha_{n-1} - \lambda\alpha_n &= \gamma_n, & \text{if } n > 0, \\ -\lambda\alpha_0 &= \gamma_0. \end{aligned}$$

As a consequence, we can sum up the coefficients to get

$$\begin{aligned}\sum_{j=1}^{n} \lambda^j \left(\alpha_j - \frac{1}{\lambda}\alpha_{j-1}\right) &= \sum_{j=1}^{n} \lambda^j \alpha_j - \sum_{j=1}^{n} \lambda^{j-1}\alpha_{j-1} \\ &= \sum_{j=1}^{n} \lambda^j \alpha_j - \sum_{j=0}^{n-1} \lambda^j \alpha_j = \lambda^n \alpha_n - \alpha_0.\end{aligned}$$

On the other hand, we use the fact that $(A^* - \lambda\,\mathrm{id})\,\alpha = \gamma$ to get

$$\alpha_n = \lambda^{-n}\left(\sum_{j=1}^{n} \lambda^j \left(\alpha_j - \frac{1}{\lambda}\alpha_{j-1}\right) + \alpha_0\right) = -\lambda^{-(n+1)} \sum_{j=0}^{n} \lambda^j \gamma_j.$$

If $|\lambda| < 1$, it is easy to see that there exist $\gamma \in \mathcal{H}$ so that $|\alpha_n| \to +\infty$ as $n \to +\infty$, and therefore $A^* - \lambda\,\mathrm{id}$ does not have an inverse from $\mathcal{H}$ to itself. As a consequence $B_1(0) \subseteq \sigma(A^*)$, and given that the spectrum is closed, we get $\overline{B_1(0)} \subseteq \sigma(A^*) \subseteq \overline{B_1(0)}$, which implies $\sigma(A^*) = \overline{B_1(0)}$.

To prove **c**, a simple computation first gives that $(B\alpha)_n = (1 - \delta_{n,0})\,\alpha_n$. From this it is easy to see that $\|B\| = 1$. B is also self-adjoint because we get $B^* = (A^*A)^* = A^*A^{**} = B$. Given that $Be^0 = 0$ and that $Be^j = e^j$ for any $j > 0$, we also get that $\{0, 1\} \subseteq B$. Given that B is self-adjoint, $\sigma(B) \subseteq \mathbb{R}$. Let now $\lambda \in \mathbb{R}\backslash\{0, 1\}$. If we consider the equation $(B - \lambda\,\mathrm{id})\,\alpha = \gamma$, we get that fixed $\gamma \in \mathcal{H}$, α needs to be

$$\begin{aligned}(1-\lambda)\,\alpha_n &= \gamma_n, & &\text{if } n > 0,\\ -\lambda\alpha_0 &= \gamma_0,\end{aligned}$$

and as a consequence we can define

$$\left((A - \lambda\,\mathrm{id})^{-1}\gamma\right)_n := \begin{cases} \frac{1}{1-\lambda}\gamma_n & \text{if } n > 0,\\ -\frac{1}{\lambda}\gamma_0 & \text{if } n = 0,\end{cases}$$

and this is a well defined linear bounded operator, implying that $\lambda \in \rho(B)$. We then conclude that $\sigma(B) = \{0, 1\}$.

□

6.4 Exercise 4 - Operator norm of multiplication for a function

Consider the interval $I = (a, b) \subseteq \mathbb{R}$ and the Hilbert space $\mathcal{H} := L^2(I)$. Consider $\varphi \in C(I)$ a real valued continuous function with $\|\varphi\|_\infty < +\infty$. Consider the operator T_φ defined for any $\psi \in \mathcal{H}$ as

$$T_\varphi \psi(x) := \varphi(x)\,\psi(x). \tag{45}$$

Prove that T_φ is a well defined linear bounded operator and prove that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

Hint: Show first that $\varphi(I) \subseteq \sigma(T_\varphi)$ and use the fact that the spectrum is closed to show that the same is true for the closures. Next, show that $\left(\overline{\sigma(T_\varphi)}\right)^{\mathrm{c}} \subseteq \rho(T_\varphi)$ to conclude.

Proof. Let $y_0 \in \varphi(I)$ and let $x_0 \in I$ such that $\varphi(x_0) = y_0$. Consider the sequence given by

$$\psi_n(x) := \begin{cases} \sqrt{n} & |x - x_0| \leqslant \frac{1}{2n}, \\ 0 & |x - x_0| > \frac{1}{2n}. \end{cases}$$

We then get that $\|\psi_n\|_2 = 1$ and therefore

$$\begin{aligned} \lim_{n\to+\infty} \frac{\|T_\varphi \psi_n - y\psi_n\|_2}{\|\psi_n\|_2} &= \lim_{n\to+\infty} \left(\sqrt{n} \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} (\varphi(x) - y)\, dx \right)^{\frac{1}{2}} \\ &= \lim_{n\to+\infty} \frac{1}{\sqrt[4]{n}} \left(n \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} \varphi(x)\, dx - y \right)^{\frac{1}{2}}. \end{aligned}$$

From the mean value theorem for integrals, given that φ is a continuous function, we get that

$$\lim_{n\to+\infty} n \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} \varphi(x)\, dx = \varphi(x_0) = y,$$

and as a consequence

$$\lim_{n\to+\infty} \frac{\|T_\varphi \psi_n - y\psi_n\|_2}{\|\psi_n\|_2} = 0.$$

As we saw in class, this implies that $y \in \sigma(T_\varphi)$; this implies that $\varphi(I) \subseteq \sigma(T_\varphi)$, and given that the spectrum is closed we get that $\overline{\varphi(I)} \subseteq \sigma(T_\varphi)$.

On the other hand, let $\lambda \notin \varphi(I)$; then the operator $(T - \lambda\, \mathrm{id})^{-1}$ is defined as

$$(T - \lambda\, \mathrm{id})^{-1} \psi(x) = \frac{1}{\varphi(x) - \lambda} \psi(x),$$

and its norm is bounded by $\left\|(T - \lambda\, \mathrm{id})^{-1}\right\| \leqslant \sup_{x\in\mathbb{R}} |\varphi(x) - \lambda|^{-1}$, which is finite by hypotheses. As a consequence we get that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

□

7 Exercise Sheet 7

7.1 Exercise 1 - Application of the UBP to the dual space

Let V be a Banach space and E a nonempty subset of V such that for any $\xi \in V^*$ there exists a finite constant C_ξ such that

$$\sup_{x \in E} |\xi(x)| \leqslant C_\xi. \tag{46}$$

Prove that E must be bounded.

Hint: Consider the map $J : V \to V^{**}$ defined as

$$[J(x)](\xi) := \xi(x) \qquad \forall x \in V,\ \xi \in V^*. \tag{47}$$

Prove that $\|J(x)\|_{V^{**}} = \|x\|$ for any $x \in V$. Use the Uniform Boundedness Principle to show that $J(E)$ is bounded and conclude.

Proof. Consider $x \in V$. Recall that we proved that for any x we have

$$\|x\| = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |\xi(x)|.$$

We then get

$$\|J(x)\|_{V^{**}} = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |[J(x)](\xi)| = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |\xi(x)| = \|x\|.$$

Consider now the set $J(E) \subseteq V^{**}$. Consider $\xi \in V^*$; using the hypothesis we then get

$$\sup_{x \in E} |[J(x)](\xi)| = \sup_{x \in E} |\xi(x)| \leqslant C_\xi.$$

We can then apply the uniform boundedness principle to get that there exists a constant C such that

$$\sup_{x \in E} \|J(x)\|_{V^{**}} \leqslant C.$$

As a consequence we get

$$\sup_{x \in E} \|x\| = \sup_{x \in E} \|J(x)\|_{V^{**}} \leqslant C,$$

and therefore E is bounded.

□

7.2 Exercise 2 - Projection valued measures

Consider (X, Ω) a measurable space (i.e., a set X with a σ-algebra Ω in it), and consider a projection-valued measure with values in $\mathcal{H}$ an Hilbert space. Let $E,\ F \in \Omega$.

a Prove that if $E \cap F = \varnothing$ then $\operatorname{Ran}\mu(E) \perp \operatorname{Ran}\mu(F)$.

b Prove that $\mu(E)\mu(F)$ is an orthogonal projector and that

$$\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F). \tag{48}$$

Proof. To prove **a** first recall that from the definition of projection-valued measure we get that for any $E,\ F \in \Omega$ we have $\mu(E \cap F) = \mu(E)\mu(F)$. Therefore if $E \cap F = \varnothing$ we have that $\mu(E)\mu(F) = \mu(F)\mu(E) = \mu(\varnothing) = 0$. Let now $\psi \in \operatorname{Ran}\mu(E)$, $\phi \in \operatorname{Ran}\mu(F)$. Given that $\mu(E)$ and $\mu(F)$ are orthogonal projectors, we get $\psi = \mu(E)\psi$ and $\phi\mu(F)\phi$, and as a consequence

$$\langle \phi, \psi \rangle = \langle \mu(F)\phi, \mu(E)\psi \rangle = \langle \phi, \mu(F)^* \mu(E)\psi \rangle = \langle \phi, \mu(F)\mu(E)\psi \rangle = 0,$$

and therefore $\operatorname{Ran}\mu(E) \perp \operatorname{Ran}\mu(F)$.

To prove **b**, first we get that in general for any $E,\ F \in \Omega$ we get $\mu(E)\mu(F) = \mu(E \cap F)$, and given that the latter is an orthogonal projector, also the former is. To prove (48), we first prove $\subseteq$. Indeed we get trivially that $\operatorname{Ran}(\mu(E)\mu(F)) \subseteq \operatorname{Ran}\mu(E)$, and on the other hand $\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}(\mu(F)\mu(E)) \subseteq \operatorname{Ran}\mu(F)$, therefore it must be included in the intersection.

On the other hand, to prove $\supseteq$ let $\psi \in \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F)$. Then we get that $\mu(E)\psi = \psi = \mu(F)\psi$. As a consequence we get $\psi = \mu(E)\psi = \mu(E)\mu(F)\psi \in \operatorname{Ran}(\mu(E)\mu(F))$, and this concludes the proof.

□

7.3 Exercise 3 - $[A, B] = 0 \Rightarrow [f(A), B] = 0$

Let $\mathcal{H}$ be an Hilbert space. Let A be a self-adjoint bounded operator over $\mathcal{H}$. Let B a bounded operator over $\mathcal{H}$ such that $[A, B] = 0$. Consider a bounded complex-valued measurable function f. Prove that $[f(A), B] = 0$.

Proof. Notice first that if $[A, B] = 0$ then $[A^n, B] = 0$ for any $n \in \mathbb{N}$. As a consequence, if f is a polynomial we also get $[f(A), B] = 0$. Consider now f a real-valued continuous function; from Weierstrass theorem we get that there exists a sequence of polynomials p_n that converges uniformly to f as n goes to infinity, and applying the result to the sequence of polynomials we get that also $f(A)$ commutes with B. Now, any complex-valued function f can be written as $f = \mathrm{Re}f + i\mathrm{Im}f$, and given that $\mathrm{Re}f$ and $\mathrm{Im}f$ are continuous and real-valued the result is also true for complex-valued continuous functions. Consider now the set $\mathcal{F} : \{f : \sigma(A) \to \mathbb{C} |\ [f(A), B]\}$; so far we proved that any complex-valued continuous function is in $\mathcal{F}$. Given that $\mathcal{F}$ is closed by uniformly bounded pointwise limit, we get that $\mathcal{F} = L^\infty(\sigma(A); \mathbb{C})$, which concludes the result.

□

7.4 Exercise 4 - Norm and spectral radius

Let $\mathcal{H}$ be an Hilbert space. Let T be a bounded operator over $\mathcal{H}$. We proved in class that in general $R(T) \leqslant \|T\|$, where

$$R(T) := \sup_{\lambda \in \sigma(T)} |\lambda| . \tag{49}$$

Exhibit an explicit operator such that $R(T) < \|T\|$.

Proof. Consider the operator T defined on the Hilbert space $\mathcal{H} := L^2(I)$, with $I = (0, 1)$ as

$$T\psi(x) := \int_0^1 \psi(x)\, dx.$$

T is a well-defined bounded linear operator and we proved in one of the exercise sessions that the spectrum of T is $\sigma(T) = \{0\}$, and therefore $R(T) = 0$. On the other hand, $T \neq 0$ implies $\|T\| > 0 = R(T)$.

□

8 Exercise Sheet 8

8.1 Exercise 1 - Commuting operators and invertibility

a Let $\mathcal{H}$ be an Hilbert space. Suppose $A, B \in \mathcal{B}(\mathcal{H})$ with $[A, B] = 0$ and A not invertible. Prove that AB is not invertible.

Hint: Prove first that if AB were invertible then A would have both a left and a right inverse. Then prove that those would need to be equal and conclude.

b Prove that if we do not assume A and B to commute, the result in **a** is false.

Proof. To prove **a**, suppose first that AB is invertible; this means that there is an operator C such that $ABC = \mathrm{id} = CAB$. Given that $[A, B] = 0$, we can also write $A(BC) = \mathrm{id} = (CB)A$. Now, to prove that $BC = CB$, given that A and B commute, we can write $BC = (CAB)BC = CB(ABC) = CB$. Therefore this implies that if AB is invertible then A is invertible, proving the result.

To prove **a** is enough to consider a counter example; consider A and A^* as in Exercise 3 in the Exercise Sheet of the 14.02.2014. We have that $[A, A^*] \neq 0$, both A and A^* are bounded and not invertible, but $AA^* = \mathrm{id}$, which is invertible.

□

8.2 Exercise 2 - An operator with a closed extension is closable

Let $\mathcal{H}$ be an Hilbert space. Let A be an unbounded linear operator on $\mathcal{H}$. Suppose there exists a closed operator C that extends the operator A. Prove that A is closable.

Proof. Recall that $G(T) := \{(\psi, T\psi) \in \mathcal{H} \times \mathcal{H} |\ \psi \in \mathcal{D}(T)\}$ is the graph of an operator T. Consider $\overline{G(A)}$; we want to prove that it corresponds to a well defined (closed) linear operator. Define the following operator:

$$\begin{aligned} \mathcal{D}(B) &:= \{\psi \in |\ \exists \varphi \in s.t. (\psi, \varphi) \in \overline{G(A)}\} \\ B &:= C|_{\mathcal{D}(B)} . \end{aligned}$$

Given that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ is dense, we get that B is densily defined. Moreover, from the linearity of C we also get that B is linear.

From the fact that C is an extension of A we get that for any $\psi \in \mathcal{D}(A)$, $B\psi = C\psi = A\psi$, so B is an extension of A. As a consequence, $G(A) \subseteq G(B)$.

On the other hand, given that C is a closed extension of A we get that $\overline{G(A)} \subseteq \overline{G(C)} = G(C)$, so if $(\psi, \varphi) \in \overline{G(A)}$ this implies $\varphi = C\psi$. On the other hand, if $(\psi, \varphi) \in \overline{G(A)}$ then $\psi \in \mathcal{D}(B)$ and therefore $B\psi = C\psi = \varphi$ and $(\psi, \varphi) \in G(B)$. Therefore we have $\overline{G(A)} \subseteq G(B)$.

Suppose now that $(\psi, B\psi) \in G(B)$. Then given that $\psi \in \mathcal{D}(B)$ there exists an element $\varphi \in \mathcal{H}$ such that $(\psi, \varphi) \in \overline{G(A)}$; but $\overline{G(A)} \subseteq G(C)$ implies $\varphi = C\psi = B\psi$, and therefore

$G(B) \subseteq \overline{G(A)}$, which together with the inclusion above shows that $G(B) = \overline{G(A)}$ and implies that A is closable.

□

8.3 Exercise 3 - Explicit norm of resolvent operator

Let $\mathcal{H}$ be an Hilbert space. Let A be self-adjoint.

a Suppose $\lambda_0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Prove that

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \frac{1}{d(\lambda_0, \sigma(A))}, \tag{50}$$

where $d(x, Y) := \inf_{y \in Y} |x - y|$, with $x \in \mathbb{C}$, $Y \subseteq C$.

Hint: Think of $(A - \lambda_0 \,\mathrm{id})^{-1}$ *as a function of* A *in the sense of the functional calculus of* A.

b Let $\lambda_0 \in \mathbb{C}$ and suppose that there exists $\varepsilon > 0$ and some nonzero $\psi \in \mathcal{H}$ such that

$$\|A\psi - \lambda_0\psi\| < \varepsilon \|\psi\|. \tag{51}$$

Prove that there exists $\lambda \in \sigma(A)$ such that $|\lambda - \lambda_0| < \varepsilon$.

Proof. Recall that there exists a projection-valued measure μ^A such that

$$A = \int_{\sigma(A)} \lambda d\mu^A(\lambda),$$

$$f(A) = \int_{\sigma(A)} f(\lambda)\, d\mu^A(\lambda).$$

Let $\lambda_0 \in \rho(A)$; given that the spectrum of A is closed, we have $d(\lambda_0, \sigma(A)) > 0$. The function $f(\lambda) := (\lambda - \lambda_0)^{-1}$ is then continuous and bounded on $\sigma(A)$, with $\sup_{\lambda \in \sigma(A)} |f(\lambda)| = d(\lambda_0, \sigma(A))^{-1}$. Now, we know that if $g(\lambda) = \lambda - \lambda_0$, on the one hand $g(A) = A - \lambda_0 \,\mathrm{id}$ and on the other hand $g(\lambda) f(\lambda) = f(\lambda) g(\lambda) = 1$. As a consequence we get that $f(A) = (A - \lambda_0 \,\mathrm{id})^{-1}$. To get (50) then we use the functional calculus to get

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \left\|\int_{\sigma(A)} f(\lambda)\, d\mu^A(\lambda)\right\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \frac{1}{d(\lambda_0, \sigma(A))}.$$

To prove **b**, assume (51); if $\lambda_0 \in \sigma(A)$, we can take $\lambda = \lambda_0$. Assume now that $\lambda_0 \in \rho(A)$. We have that

$$\left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| \geqslant \frac{\left\|(A - \lambda_0 \,\mathrm{id})^{-1} (A - \lambda_0 \,\mathrm{id})\, \psi\right\|}{\|(A - \lambda_0 \,\mathrm{id})\, \psi\|} = \frac{\|\psi\|}{\|(A - \lambda_0 \,\mathrm{id})\, \psi\|} > \frac{1}{\varepsilon}.$$

Using then (50) we get

$$\frac{1}{\varepsilon} < \left\|(A - \lambda_0 \,\mathrm{id})^{-1}\right\| = \frac{1}{d(\lambda_0, \sigma(A))},$$

which concludes the proof.

□

8.4 Exercise 4 - The delta is not a closable operator

Let $\mathcal{H} = L^2(I)$, with $I = [0,1]$. Consider the operator A with domain $D(A) = C(I)$ and with action

$$A\psi(x) = \psi(0), \qquad \forall \psi \in D(A). \tag{52}$$

Prove that A is not closable.

Proof. Consider the graph of A given as $G(A) = \{(\psi, \psi(0)) \mid \psi \in C(I)\}$; considering $\psi = 0$, we get that $(0,0) \in G(A)$.

Moreover, let ψ_n be a sequence of continuous functions with $\psi(I) \in [0,1]$, $\psi(x) = 0$ for any $x \in (1/n, 1]$ and $\psi(x) = 1$ for any $x \in [0, 1/(2n))$.

Then given that $\|\psi\| \leqslant 1/n$, we get $\psi_n \to 0$ in $\mathcal{H}$ as $n \to +\infty$; on the other hand, we have that $A\psi_n(x) = 1$ for any x and for any n, so $A\psi_n \to 1$ in $\mathcal{H}$ as $n \to +\infty$. As a consequence, $(0,1) \in \overline{G(A)}$, which implies that A is not closable.

□

9 Exercise Sheet 9

9.1 Exercise 1 - Hardy inequality

Let $k \in \mathbb{Z}$, $d \in \mathbb{N}$, $k + d \neq 0$. Let D be defined as

$$D := \begin{cases} C_c^\infty\left(\mathbb{R}^d\right) & \text{if } k \geqslant 0, \\ C_c^\infty\left(\mathbb{R}^d \setminus \{\mathbf{0}\}\right) & \text{if } k \leqslant -1, \ k + d \neq 0. \end{cases} \tag{53}$$

Prove that for any $\psi \in D$

$$\int_{\mathbb{R}^d} |\mathbf{x}|^k |\psi(\mathbf{x})|^2 \, d\mathbf{x} \leqslant \frac{4}{|k+d|^2} \int_{\mathbb{R}^d} |\mathbf{x}|^{k+2} |\nabla\psi(\mathbf{x})|^2 \, d\mathbf{x}. \tag{54}$$

Hint: Use the fact that

$$|\mathbf{x}|^k = \frac{1}{k+d} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(|\mathbf{x}|^k x_j\right) \tag{55}$$

to integrate by part on the left hand side of (54) *and then use the Cauchy-Schwartz inequality.*

Remark: Notice that in particular if $k = -2$ (and $d \neq 2$) this implies that as operators

$$\frac{1}{|\mathbf{x}|^2} \leqslant -\frac{4}{|d-2|}\Delta. \tag{56}$$

A generalisation of this formula is called in the literature the **Hardy inequality**.

Proof. We will use the shorthand notation of div for a divergence of a vector field, meaning that if $\mathbf{F}$ is a vector field on $\mathbb{R}^d$, we define

$$\operatorname{div} \mathbf{F}(\mathbf{x}) := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} F_j(\mathbf{x}).$$

With this notation in mind we have that the Green theorem can be written as

$$\int_{\mathbb{R}^d} \operatorname{div} F(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x} = -\int_{\mathbb{R}^d} F \cdot \nabla g(\mathbf{x})\, d\mathbf{x},$$

and we can write $|\mathbf{x}|^k = (k+d)^{-1} \operatorname{div}\left(|\mathbf{x}|^k \mathbf{x}\right)$.

Let $\psi \in D$ and consider the left-hand side of (54); we get

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbf{x}|^k |\psi(\mathbf{x})|^2 d\mathbf{x} &= \frac{1}{k+d}\int_{\mathbb{R}^d} \operatorname{div}\left(|\mathbf{x}|^k x\right)|\psi(\mathbf{x})|^2 d\mathbf{x} \\ &= -\frac{1}{k+d}\int_{\mathbb{R}^d} |\mathbf{x}|^k x\cdot\nabla\left(|\psi(\mathbf{x})|^2\right) d\mathbf{x} \\ &= -\frac{2}{k+d}\int_{\mathbb{R}^d} |\mathbf{x}|^k x\cdot \operatorname{Re}\left(\overline{\psi(\mathbf{x})}\nabla\psi(\mathbf{x})\right) d\mathbf{x} \\ &\leqslant \frac{2}{|k+d|}\int_{\mathbb{R}^d} |\mathbf{x}|^{k+1}|\psi(\mathbf{x})||\nabla\psi(\mathbf{x})| d\mathbf{x} \\ &\leqslant \frac{2}{|k+d|}\left(\int_{\mathbb{R}^d}|\mathbf{x}|^{2(k+1-\eta)}|\psi(\mathbf{x})|^2 d\mathbf{x}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\mathbf{x}|^{2\eta}|\nabla\psi(\mathbf{x})| d\mathbf{x}\right)^{\frac{1}{2}}.\end{aligned}$$

If we choose $\eta = \frac{k+2}{2}$ we get

$$\int_{\mathbb{R}^d} |\mathbf{x}|^k |\psi(\mathbf{x})|^2 d\mathbf{x} \leqslant \frac{4}{|k+d|^2}\int_{\mathbb{R}^d}|\mathbf{x}|^{k+2}|\nabla\psi(\mathbf{x})| d\mathbf{x}.$$

□

9.2 Exercise 2 - The Coulomb hamiltonian is self-adjoint

a Let $\mathcal{H} := L^2(\mathbb{R}^3)$. Define (as in class) the operator H_0 with[5]

$$\mathcal{D}(H_0) := H^2(\mathbb{R}^3) \equiv \left\{\psi\in\mathcal{H} \mid |\mathbf{k}|^2\widehat{\psi}(\mathbf{k}) \in L^2(\mathbb{R}^3)\right\}, \tag{57}$$

$$H_0\psi = -\Delta\psi = \left(|\mathbf{k}|^2\widehat{\psi}(\mathbf{k})\right)^{\vee}, \qquad \forall\psi\in\mathcal{D}(H_0). \tag{58}$$

Prove that H_0 is closed.

b Let $\mathcal{D}(H) := \mathcal{D}(H_0)$. Define $H := H_0 + \frac{1}{|\mathbf{x}|}$. Prove that H is well-defined and closed. (Assume, if necessary, to know that there exists a positive constant C such that for any $\psi \in H^2(\mathbb{R}^3)$ it holds $\|\psi\|_{L^\infty} \leqslant C\|\psi\|_{H^2}$).

Hint: Use the fact that $H^2(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3)$ to prove that is well-defined. To prove the closure, use (54) from Exercise 1 to show and subsequently use that $\forall\varepsilon > 0$, $\forall\psi\in\mathcal{D}(H)$

$$\left\|\frac{1}{|\mathbf{x}|}\psi\right\|_{L^2} \leqslant \frac{2}{\varepsilon}\|\psi\|_{L^2} + \varepsilon\|H_0\psi\|_{L^2} \tag{59}$$

to get that

$$\|H_0\psi\|_{L^2} \leqslant \frac{2}{\varepsilon(1-\varepsilon)}\|\psi\|_{L^2} + \frac{1}{1-\varepsilon}\|H\psi\|_{L^2}. \tag{60}$$

c Prove that H is symmetric.

[5]Recall that we proved in the exercise session that if $\|\psi\|_{H^2} := \left\|\left(1+|\mathbf{k}|^2\right)\widehat{\psi}\right\|_{L^2}$, then $H^2(\mathbb{R}^3)$ is closed with respect to $\|\cdot\|_{H^2}$.

d Prove that H is self-adjoint.

Hint: Use the fact that $\frac{1}{|x|}$ is a self-adjoint operator and apply the Kato-Rellich theorem.

Proof. Recall that we proved in the exercise session that $H^2\left(\mathbb{R}^3\right)$ is closed with respect to $\|\cdot\|_{H^2}$. To prove **a**, then, consider a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(H_0)$ such that $\psi_n \to \psi$ and $H_0\psi_n \to \phi$ in $\mathcal{H}$. As a consequence we get that $\{\psi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{H^2}$ and therefore $\psi \in H^2\left(\mathbb{R}^3\right) = \mathcal{D}(H_0)$ and $H_0\psi = \varphi$, and hence H_0 is closed.

To prove **b** we first prove that H is well defined. Given that $\psi \in H^2\left(\mathbb{R}^3\right) \subseteq L^\infty\left(\mathbb{R}^3\right)$, we get that

$$\begin{aligned}\left\|\frac{1}{|\mathbf{x}|}\psi\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}|^2}|\psi(\mathbf{x})|^2\,dx \leqslant \|\psi\|_{L^\infty}\int_{B_1(\mathbf{0})}\frac{1}{|\mathbf{x}|^2}d\mathbf{x} + \int_{\mathbb{R}^3\setminus B_1(\mathbf{0})}|\psi(\mathbf{x})|^2\,d\mathbf{x}\\ &\leqslant 4\pi\|\psi\|_{L^\infty} + \|\psi\|_{L^2} \leqslant (4\pi C+1)\|\psi\|_{H^2}.\end{aligned}$$

We then use Hardy inequality and the fact that for any $\eta > 0$ we have $|\mathbf{k}|^2 \leqslant 1/\eta + \eta/4\,|\mathbf{k}|^4$, to obtain for any $\psi \in C_c^\infty\left(\mathbb{R}^3\setminus\{\mathbf{0}\}\right)$ that

$$\begin{aligned}\left\|\frac{1}{|\mathbf{x}|}\psi\right\|_{L^2}^2 &\leqslant 4\|\nabla\psi\|_{L^2}^2 = 4\int_{\mathbb{R}^3}|\mathbf{k}|^2\left|\widehat{\psi}(\mathbf{k})\right|^2 d\mathbf{x} \leqslant \frac{4}{\eta}\|\psi\|_{L^2}^2 + \eta\|H_0\psi\|_{L^2}^2\\ &\leqslant \left(\frac{2}{\sqrt{\eta}}\|\psi\|_{L^2} + \sqrt{\eta}\|H_0\psi\|_{L^2}\right)^2.\end{aligned}$$

Calling $\eta = \varepsilon^2$ we obtain (59). As a consequence we get for any $\psi \in C_c^\infty\left(\mathbb{R}^3\setminus\{\mathbf{0}\}\right)$

$$\|H_0\psi\|_{L^2} \leqslant \|H\psi\|_{L^2} + \left\|\frac{1}{|\mathbf{x}|}\psi\right\|_{L^2} \leqslant \|H\psi\|_{L^2} + \frac{2}{\varepsilon}\|\psi\|_{L^2} + \varepsilon\|H_0\|_{L^2}.$$

Choosing $\varepsilon < 1$ and collecting the identical terms on the left we obtain (60).

Suppose $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}(H)$ and that $\psi_n \to \psi$ and $H\psi_n \to \varphi$ in $\mathcal{H}$; then the sequences $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{H\psi_N\}_{n\in\mathbb{N}}$ are Cauchy sequences and using (60) we get that also $\{H_0\psi_n\}_{n\in\mathbb{N}}$ is. From **a** we then get that $\psi \in \mathcal{D}(H_0) = \mathcal{D}(H)$ and that $H_0\psi_n \to H_0\psi$. Moreover we get that

$$\begin{aligned}\lim_{n\to+\infty}\left\|\frac{1}{|\mathbf{x}|}(\psi_n-\psi)\right\|_{L^2} &\leqslant \lim_{n\to+\infty}(4\pi C+1)\|\psi_n-\psi\|_{H^2}\\ &= \lim_{n\to+\infty}(4\pi C+1)\sqrt{\|\psi_n-\psi\|_{L^2}^2 + \|H_0(\psi_n-\psi)\|_{L^2}^2} = 0,\end{aligned}$$

and as a consequence $H\psi_n \to H\psi$, so H is closed.

To prove **c**, consider $\psi,\ \varphi \in \mathcal{D}(H) = H^2\left(\mathbb{R}^3\right)$; then we get

$$\langle\psi, H^*\varphi\rangle = \langle H\psi,\varphi\rangle = \langle -\Delta\psi,\varphi\rangle + \langle\frac{1}{|\mathbf{x}|}\psi,\varphi\rangle.$$

We already showed in class that $-\Delta$ is symmetric, so we get that

$$\langle\psi, H^*\varphi\rangle = \langle\psi,-\Delta\varphi\rangle + \langle\psi,\frac{1}{|\mathbf{x}|}\varphi\rangle = \langle\psi,H\varphi\rangle,$$

and therefore H is symmetric.

To prove **d** notice that if we define the operator V as the operator given by

$$\mathcal{D}(V) := \left\{\psi \in \mathcal{H} \middle| \frac{1}{|x|}\psi(x) \in \mathcal{H}\right\}$$
$$V\psi(x) := \frac{1}{|x|}\psi(x),$$

this is a well defined self-adjoint operator. Indeed it is trivially symmetric, and therefore V^* is an extension of V. Furthermore, let ψ in $\mathcal{D}(V^*)$ and consider $\phi \in \mathcal{S}(\mathbb{R}^3)$ the space of Schwartz functions. In particular $\phi \in \mathcal{D}(V)$, and we get

$$|\langle \psi, V\phi\rangle| \leqslant C_\psi \|\phi\|_{L^2}.$$

As a consequence, using Riesz theorem, there exists an element $\xi \in L^2(\mathbb{R}^3)$ such that $\langle \xi, \phi\rangle = \langle \psi, V\phi\rangle$ for any $\phi \in \mathcal{S}(\mathbb{R}^3)$. This in particular implies that $V\psi = \xi$ almost everywhere, and therefore $V\psi \in L^2(\mathbb{R}^3)$. By the definition of the domain of V we get $\psi \in \mathcal{D}(V)$ and V is self-adjoint.

Now, choosing $\varepsilon < 1$ we can use (59) to first get that $\mathcal{D}(H_0) \subseteq \mathcal{D}(V)$. We are then in the hypothesis of the Kato-Rellich theorem, and we can conclude that $H = H_0 + V$ is self-adjoint.

□

9.3 Exercise 3 - The square root is monotonous

Let $\mathcal{H}$ an Hilbert space and let $A,\ B \in \mathcal{B}(\mathcal{H})$, $A^* = A$, $B^* = B$

- **a** Suppose[6] $A \geqslant \mathrm{id}$; prove that A is invertible with $A^{-1} \in \mathcal{B}(\mathcal{H})$ and that $0 \leqslant A^{-1} \leqslant \mathrm{id}$.
- **b** Suppose $0 \leqslant A \leqslant B$; prove that for any $\lambda > 0$, $A+\lambda\,\mathrm{id}$ and $B+\lambda\,\mathrm{id}$ are invertible with $(A+\lambda\,\mathrm{id})^{-1}$, $(B+\lambda\,\mathrm{id})^{-1} \in \mathcal{B}(\mathcal{H})$ and that we have $(B+\lambda\,\mathrm{id})^{-1} \leqslant (A+\lambda\,\mathrm{id})^{-1}$.
- **c** Suppose $0 \leqslant A \leqslant B$; prove that $\sqrt{A} \leqslant \sqrt{B}$.

  Hint: Prove and use the fact that

$$\sqrt{x} = \frac{1}{\pi}\int_0^{+\infty} \frac{1}{\sqrt{\lambda}}\left(1 - \frac{\lambda}{x+\lambda}\right) d\lambda, \qquad \forall x \geqslant 0. \tag{61}$$

Proof. To prove **a** we first notice that $A \geqslant \mathrm{id}$ implies that $\sigma(A) \subseteq [1, +\infty)$, and therefore $0 \notin \sigma(A)$. By definition of spectrum this implies that $A^{-1} \in \mathcal{B}(\mathcal{H})$. Using functional calculus, if μ is the spectral measure associated to A, for any $\psi \in \mathcal{H}$ we get

$$\langle \psi, A^{-1}\psi\rangle = \langle \psi, \int_{\sigma(A)} \frac{1}{\lambda} d\mu(\lambda)\,\psi\rangle \leqslant \sup_{\lambda \in \sigma(A)} \frac{1}{\lambda}\langle \psi, \int_{\sigma(A)} d\mu(\lambda)\,\psi\rangle \leqslant \|\psi\|^2.$$

[6]Recall that $A \geqslant 0$ if for any $\psi \in \mathcal{D}(A)$, $\langle \psi, A\psi\rangle \geqslant 0$ and that $A \geqslant B$ if $A - B \geqslant 0$.

Proceeding analogously we also get

$$\langle\psi, A^{-1}\psi\rangle = \langle\psi, \int_{\sigma(A)} \frac{1}{\lambda} d\mu(\lambda)\,\psi\rangle \geqslant \inf_{\lambda\in\sigma(A)} \frac{1}{\lambda}\langle\psi, \int_{\sigma(A)} d\mu(\lambda)\,\psi\rangle \geqslant 0.$$

Those chains of inequalities imply that $0 \leqslant A^{-1} \leqslant \mathrm{id}$.

To prove **b** consider $\lambda > 0$; given that $\lambda > 0$, we have

$$B + \lambda\,\mathrm{id} \geqslant A + \lambda\,\mathrm{id} \Rightarrow (A + \lambda\,\mathrm{id})^{-\frac{1}{2}} (B + \lambda\,\mathrm{id}) (A + \lambda\,\mathrm{id})^{-\frac{1}{2}} \geqslant \mathrm{id},$$

where we used the fact that $A + \lambda\,\mathrm{id} \geqslant \lambda\,\mathrm{id}$ and that $(\cdot)^{-\frac{1}{2}}$ is continuous and bounded on $[\lambda, +\infty)$ to define $(A + \lambda\,\mathrm{id})^{-\frac{1}{2}}$.

Using **a** we then get that

$$\begin{aligned}\mathrm{id} &\geqslant \left[(A + \lambda\,\mathrm{id})^{-\frac{1}{2}} (B + \lambda\,\mathrm{id}) (A + \lambda\,\mathrm{id})^{-\frac{1}{2}}\right]^{-1} \\ &= (A + \lambda\,\mathrm{id})^{\frac{1}{2}} (B + \lambda\,\mathrm{id})^{-1} (A + \lambda\,\mathrm{id})^{\frac{1}{2}}.\end{aligned}$$

Multiplying both sides from left and right by $(A + \lambda\,\mathrm{id})^{-\frac{1}{2}}$ we can conclude.

To prove **c**, we first prove (61); we get

$$\begin{aligned}\int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \left(1 - \frac{\lambda}{x + \lambda}\right) d\lambda &= \int_0^{+\infty} \frac{x}{\sqrt{\lambda}\,(x + \lambda)} d\lambda = \sqrt{x} \int_0^{+\infty} \frac{1}{\sqrt{\lambda}\,(1 + \lambda)} d\lambda \\ &= \sqrt{x} \left[2 \arctan \sqrt{\lambda}\right]_0^{+\infty} = \pi\sqrt{x}.\end{aligned}$$

As a consequence we can write for any $\psi \in \mathcal{H}$

$$\langle\psi, \sqrt{A}\psi\rangle = \langle\psi, \int_{\sigma(A)} \sqrt{\lambda} d\mu(\lambda)\,\psi\rangle = \langle\psi, \int_{\sigma(A)} \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} \left(1 - \frac{t}{t + \lambda}\right) dt d\mu(\lambda)\,\psi\rangle.$$

Now, given that $\frac{1}{\sqrt{t}}\left(1 - \frac{t}{t+\lambda}\right) \leqslant \frac{1}{\sqrt{t}}\left(1 - \frac{t}{t+\|A\|}\right) = \frac{\|A\|}{\sqrt{t}(t+\|A\|)}$ is integrable in $\sigma(A) \times [0, +\infty)$ with the measure given by the product of the spectral measure of A and the Lebesgue measure, we can exchange the order of the two integrals to get

$$\begin{aligned}\langle\psi, \sqrt{A}\psi\rangle &= \langle\psi, \frac{1}{\pi} \int_0^{+\infty} \int_{\sigma(A)} \frac{1}{\sqrt{t}} \left(1 - \frac{t}{t + \lambda}\right) d\mu(\lambda)\, dt\psi\rangle \\ &= \langle\psi, \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} \left(1 - t\,(A + t\,\mathrm{id})^{-1}\right) dt\psi\rangle.\end{aligned}$$

Using now **b** we get that for any $\psi \in \mathcal{H}$

$$\begin{aligned}\langle\psi, \sqrt{A}\psi\rangle &= \langle\psi, \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} \left(1 - t\,(A + t\,\mathrm{id})^{-1}\right) dt\psi\rangle \\ &\leqslant \langle\psi, \frac{1}{\pi} \int_0^{+\infty} \frac{1}{\sqrt{t}} \left(1 - t\,(B + t\,\mathrm{id})^{-1}\right) dt\psi\rangle = \langle\psi, \sqrt{B}\psi\rangle,\end{aligned}$$

which allows us to conclude.

□

9.4 Exercise 4 - Exercise on norm of the resolvent

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear self-adjoint operator on $\mathcal{H}$ with $A \geqslant 0$ and $\lambda > 0$. Denote with $\|\cdot\|$ the operator norm and with $\|\cdot\|_{\mathcal{H}}$ the norm induced by the inner product in the Hilbert space $\mathcal{H}$.

a Prove that $\left\|(A + \lambda \,\mathrm{id})^{-1}\right\| \leqslant 1/\lambda$.

b Prove that for all $\psi \in \mathcal{H}$,

$$\|\psi\|_{\mathcal{H}}^2 \geqslant \left\|A\,(A + \lambda \,\mathrm{id})^{-1}\,\psi\right\|_{\mathcal{H}}^2 + \lambda^2 \left\|(A + \lambda \,\mathrm{id})^{-1}\,\psi\right\|_{\mathcal{H}}^2 . \tag{62}$$

Conclude that $\left\|A\,(A + \lambda \,\mathrm{id})^{-1}\right\| \leqslant 1$.

Proof. To prove **a** recall that we proved in class that if T is a self-adjoint operator and f is a continuous and bounded function we have $\|f\,(T)\| \leqslant \sup_{\zeta \in \sigma(T)} |f\,(\zeta)|$. Moreover, we also saw that if $A \geqslant 0$ then $\sigma\,(A) \subseteq [0, +\infty)$. As a consequence we get

$$\left\|(A + \lambda \,\mathrm{id})^{-1}\right\| \leqslant \sup_{\zeta \in \sigma(A)} \frac{1}{|\zeta + \lambda|} \leqslant \sup_{\zeta \in [0,+\infty)} \frac{1}{\zeta + \lambda} \leqslant \frac{1}{\lambda}.$$

To prove **b** we get that for any $\psi \in \mathcal{H}$

$$\langle \psi, (A + \lambda \,\mathrm{id})^{-1}\,A\,(A + \lambda \,\mathrm{id})^{-1}\,\psi \rangle = \langle (A + \lambda \,\mathrm{id})^{-1}\,\psi, A\,(A + \lambda \,\mathrm{id})^{-1}\,\psi \rangle \geqslant 0.$$

As a consequence we get that

$$\begin{aligned}
&\left\|A\,(A + \lambda \,\mathrm{id})^{-1}\,\psi\right\|_{\mathcal{H}}^2 + \lambda^2 \left\|(A + \lambda \,\mathrm{id})^{-1}\,\psi\right\|_{\mathcal{H}}^2 = \\
&\quad = \langle \psi, (A + \lambda \,\mathrm{id})^{-1}\left(A^2 + \lambda^2\right)(A + \lambda \,\mathrm{id})^{-1}\,\psi \rangle \\
&\quad \leqslant \langle \psi, (A + \lambda \,\mathrm{id})^{-1}\left(A^2 + 2\lambda A + \lambda^2\right)(A + \lambda \,\mathrm{id})^{-1}\,\psi \rangle = \|\psi\|_{\mathcal{H}}^2 .
\end{aligned}$$

As a consequence we then get $\left\|A\,(A + \lambda \,\mathrm{id})^{-1}\,\psi\right\|_{\mathcal{H}} \leqslant \|\psi\|_{\mathcal{H}}$ which allows us to conclude that $\left\|A\,(A + \lambda \,\mathrm{id})^{-1}\right\| \leqslant 1$.

□

10 Exercise Sheet 10

10.1 Exercise 1 - The generator of the translation is the momentum

Let $\mathcal{H} := L^2(\mathbb{R})$ and $P := -i\partial_x$ the momentum operator defined on the domain $\mathcal{D}(P) := H^1(\mathbb{R})$ as $P\psi(x) = -i\frac{\partial\psi}{\partial x}(x)$. Consider for any $\lambda \in \mathbb{R}$ the bounded operator T_λ defined for any $\psi \in \mathcal{H}$ as $T_\lambda\psi(x) = \psi(x-\lambda)$.

Prove that $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group and that

$$T_\lambda = e^{i\lambda P} = e^{\lambda\partial_x}. \tag{63}$$

Proof. By definition it follows that if $\lambda = 0$ then for any $\psi \in \mathcal{H}$ we get $T_0\psi(x) = \psi(x)$ and therefore $T_0 = \mathrm{id}$. On the other hand, let $\lambda,\ \mu \in \mathbb{R}$; then for any $\psi \in \mathbb{H}$ we get $T_\lambda T_\mu\psi(x) = T_\mu\psi(x-\lambda) = \psi(x-\lambda-\mu) = T_{\lambda+\mu}(x)$. Consider now $\lambda \in \mathbb{R}$, $\psi,\ \varphi \in \mathcal{H}$; to prove that T_λ is a unitary operator we compute T_λ^* to get

$$\langle\psi, T_\lambda^*\varphi\rangle = \langle T_\lambda\psi, \varphi\rangle = \int_{\mathbb{R}} \overline{\psi(x-\lambda)}\varphi(x)\,dx = \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x+\lambda)\,dx = \langle\psi, T_{-\lambda}\varphi\rangle,$$

and as a consequence $T_\lambda^* = T_{-\lambda}$; therefore we get $T_\lambda T_\lambda^* = T_\lambda T_{-\lambda} = T_0 = \mathrm{id}$, $T_\lambda^* T_\lambda = T_{-\lambda}T_\lambda = T_0 = \mathrm{id}$ and for any $\lambda \in \mathbb{R}$, T_λ is unitary.

To prove that it is continuous, first consider $\varphi \in C_c^\infty(\mathbb{R})$. Then we get that there exists $R > 0$ such that for any $|\lambda| \leqslant \delta$, $\operatorname{supp}\varphi \cup \operatorname{supp} T_\lambda\varphi \subseteq B_R(0)$. As a consequence

$$|\varphi(x) - T_\lambda\varphi(x)|^2 \leqslant 4\,\|\varphi\|_{L^\infty}^2\,\chi_{B_R(0)}(x) \in L^1(\mathbb{R}).$$

We can then apply the dominated convergence theorem to get (with $\widetilde{\lambda} = \lambda - \mu$)

$$\begin{aligned}
\lim_{\lambda\to\mu} \|T_\lambda\varphi - T_\mu\varphi\|_{L^2}^2 &= \lim_{\widetilde{\lambda}\to 0} \left\|T_{\widetilde{\lambda}}\varphi - \varphi\right\|_{L^2}^2 = \lim_{\widetilde{\lambda}\to 0} \int_{\mathbb{R}} |\varphi(x) - T_\lambda\varphi(x)|^2\,dx \\
&= \int_{\mathbb{R}} \lim_{\widetilde{\lambda}\to 0} |\varphi(x) - T_\lambda\varphi(x)|^2\,dx = 0.
\end{aligned}$$

Let now $\varepsilon > 0$ and for any $\psi \in \mathcal{H}$ let $\varphi \in C_c^\infty(\mathbb{R})$ such that $\|\psi - \varphi\|_{L^2}^2 < \varepsilon$. We then get

$$\begin{aligned}
\lim_{\lambda\to\mu} \|T_\lambda\psi - T_\mu\psi\|_{L^2} &\leqslant \lim_{\lambda\to\mu} \left[\|T_\lambda\psi - T_\lambda\varphi\|_{L^2} + \|T_\lambda\varphi - T_\mu\varphi\|_{L^2} + \|T_\mu\psi - T_\mu\varphi\|_{L^2}\right] \\
&= \lim_{\lambda\to\mu} \left[2\,\|\psi - \varphi\|_{L^2} + \|T_\lambda\varphi - T_\mu\varphi\|_{L^2}\right] = 2\,\|\psi - \varphi\|_{L^2} < 2\varepsilon.
\end{aligned}$$

Sending ε to zero we get the conclusion.

Consider now A the infinitesimal generator of $\{T_\lambda\}_{\lambda\in\mathbb{R}}$. Suppose $\psi \in \mathcal{H}$; recall that if $\mathcal{F}$ represent the Fourier transform, we get

$$\mathcal{F}T_\lambda\psi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx}\psi(x-\lambda)\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ik(x+\lambda)}\psi(x)\,dx = e^{-i\lambda k}\widehat{\psi}(k).$$

Suppose now that $\psi \in H^1(\mathbb{R})$; then we get for any λ

$$\mathcal{F}\left(\frac{T_\lambda - \mathrm{id}}{\lambda}\psi\right)(k) = \frac{e^{-i\lambda k}-1}{\lambda}\widehat{\psi}(k).$$

Given that $\mathcal{F}P\psi(k) = k\widehat{\psi}(k)$, we get

$$\left\|\frac{T_\lambda - \mathrm{id}}{\lambda}\psi - iP\psi\right\|_2^2 = \left\|\frac{e^{-i\lambda k}-1}{\lambda}\widehat{\psi} - ik\widehat{\psi}\right\|_2^2 = \int_{\mathbb{R}}\left|\frac{e^{-i\lambda k}-1}{\lambda} - ik\right|^2 |\psi(k)|^2\, dk$$

Given that $\left|\frac{e^{-i\lambda k}-1}{\lambda}\right|^2 \leqslant |k|^2$, and $|k|^2\left|\widehat{\psi}(k)\right|^2 \in L^1(\mathbb{R})$, we can conclude

$$\lim_{\lambda\to 0}\left\|\frac{T_\lambda - \mathrm{id}}{\lambda}\psi - iP\psi\right\|_2^2 = \int_{\mathbb{R}}\lim_{\lambda\to 0}\left|\frac{e^{-i\lambda k}-1}{\lambda} - ik\right|^2 |\psi(k)|^2\, dk = 0.$$

So clearly $H^1(\mathbb{R}) \subseteq \mathcal{D}(A)$ and A is an extension of P. Now, given that as a consequence we have that P^* is an extension of A^* and that both A and P are self-adjoint, we get that $A = P$; therefore by Stone theorem $e^{\lambda\partial_x} = e^{i\lambda P} = T_\lambda$.

□

10.2 Exercise 2 - Condition for self-adjointness (complement to the class)

Let $\mathcal{H}$ be an Hilbert space, A a symmetric operator and $\mu > 0$ a positive real number. Prove that the following are equivalent.

- **a** A is self-adjoint.
- **b** $\mathrm{Ran}(A + i\mu\,\mathrm{id}) = \mathrm{Ran}(A - i\mu\,\mathrm{id}) = \mathcal{H}$.

Proof. In class we proved that if A is symmetric, $A = A^*$ if and only if $\mathrm{Ran}(A \pm i\,\mathrm{id}) = \mathcal{H}$. Now, A is symmetric if and only if $\frac{1}{\mu}A$ is, and given that $A + i\mu\,\mathrm{id} = \mu\left(\frac{1}{\mu}A + i\,\mathrm{id}\right)$, $\mathrm{Ran}(A + i\mu\,\mathrm{id}) = \mathrm{Ran}\left(\frac{1}{\mu}A + i\,\mathrm{id}\right)$, the result follows from the result proven in class.

□

10.3 Exercise 3 - Unitary operators as exponentials

Let $\mathcal{H}$ be an Hilbert space. Let $U \in \mathcal{B}(\mathcal{H})$. Prove that U is unitary if and only if there exist a self-adjoint operator A on $\mathcal{H}$ such that $U = e^{iA}$.

Proof. Recall that from functional calculus for self-adjoint operators we have $f(A)^* = \overline{f}(A)$. Then, given that $\overline{e^{ix}} = e^{-ix}$, we get $\left(e^{iA}\right)^* = e^{-iA}$. As a consequence we get $e^{iA}\left(e^{iA}\right)^* = e^{iA}e^{-iA} = \mathrm{id} = e^{-iA}e^{iA} = \left(e^{iA}\right)^* e^{iA}$, and therefore $U = e^{iA}$ is a unitary operator.

Suppose now U is unitary. Then we get $\sigma(U) \subseteq \overline{B_1(0)}$ because $\|U\| = 1$. On the other hand we get that if $\lambda \in B_1(0)$, then $U - \lambda \operatorname{id} = U(\operatorname{id} - \lambda U^*) = (\operatorname{id} - \lambda U^*) U$, and given that $\|\lambda U^*\| = |\lambda| < 1$ and U is unitary, then $U - \lambda \operatorname{id}$ is invertible and $\lambda \notin \sigma(U)$. Therefore $\sigma(U) \subseteq \mathbb{S}_1$ (where $\mathbb{S}_1 = \{\psi \in \mathcal{H} | \ \|\psi\| = 1\}$).

Now, the map x^t is bounded from $\sigma(U) \to \mathbb{C}$ for any $t \in \mathbb{R}$. Define then $U(t) := U^t$, defined through the functional calculus for normal operators. By construction we have that $U(t)$ is a strongly continuous one-parameter unitary group, so let A be the self-adjoint infinitesimal generator. As a consequence of Stone theorem, we get that $U(t) = e^{itA}$, and therefore, $U = U(1) = e^{iA}$.

□

10.4 Exercise 4 - Bogoliubov diagonalization - part I

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \operatorname{id}, \tag{64}$$

$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{65}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^* A_+ + A_-^* A_-\right) + \zeta\left(A_+^* A_-^* + A_+ A_-\right). \tag{66}$$

a Prove that H is self-adjoint.

b Prove that there exist operators $C_\pm$ and numbers $\alpha,\ \beta \in \mathbb{R}$ such that

$$\left[C_\pm, C_\pm^*\right] = \operatorname{id}, \tag{67}$$

$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{68}$$

$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta. \tag{69}$$

Hint: Define

$$C_\pm := \gamma_\pm A_\pm + \xi_\pm A_\mp^* \tag{70}$$

for some $\gamma_\pm,\ \xi_\pm \in \mathbb{R}$*. Use* (67) *and* (68) *to deduce that* $\gamma_+ = \gamma_-$*,* $\xi_+ = \xi_-$ *and that* $\gamma_\pm^2 - \xi_\pm^2 = 1$*. Calculate* $C_\pm^* C_\pm$ *and deduce* (69)*.*

Proof. To prove **a**, given that $A_\pm$ are bounded operators, first notice that $\left(A_\pm^* A_\pm\right)^* = A_\pm^* A_\pm^{**} = A_\pm^* A_\pm$. On the other hand $\left(A_+^* A_-^*\right)^* = A_-^{**} A_+^{**} = A_- A_+ = A_+ A_-$, and therefore $\left(A_+^* A_-^* + A_+ A_-\right)^* = A_+^* A_-^* + A_+ A_-$. As a consequence H is self-adjoint.

To prove **b** , consider $C_\pm$ defined as (70). Using (67) we get

$$\begin{aligned}\operatorname{id} = \left[C_\pm, C_\pm^*\right] &= \left[\gamma_\pm A_\pm + \xi_\pm A_\mp^*, \gamma_\pm A_\pm^* + \xi_\pm A_\mp\right] \\ &= \gamma_\pm^2\left[A_\pm, A_\pm^*\right] + \xi_\pm^2\left[A_\mp^*, A_\mp\right] = \left(\gamma_\pm^2 - \xi_\pm^2\right)\operatorname{id}.\end{aligned}$$

Now, given that the function sinh is bijective, let $\theta_\pm \in \mathbb{R}$ such that $\xi_\pm = \sinh(\theta_\pm)$. Then $\gamma_\pm^2 = 1 + \xi_\pm^2 = 1 + \sinh^2(\theta_\pm) = \cosh(\theta_\pm)$. Using (68) we then get

$$\begin{aligned}0 &= [C_+, C_-] = \left[\gamma_+ A_+ + \xi_+ A_-^*, \gamma_- A_- + \xi_- A_+^*\right] \\ &= \gamma_+ \xi_- \left[A_+, A_+^*\right] + \xi_+ \gamma_- \left[A_-^*, A_-\right] = (\gamma_+ \xi_- - \xi_+ \gamma_-)\, \mathrm{id} \\ &= (\cosh(\theta_+) \sinh(\theta_-) - \cosh(\theta_-) \sinh(\theta_+))\, \mathrm{id} = \sinh(\theta_- - \theta_+)\, \mathrm{id}\,.\end{aligned}$$

From the fact that $\sinh^{-1}(0) = 0$, we get that $\theta_+ = \theta_- = \theta$. We then got that $C_\pm = \cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*$, and we now consider $C_\pm^* C_\pm$:

$$\begin{aligned}C_\pm^* C_\pm &= \left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right)^* \left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right) \\ &= \left(\cosh(\theta) A_\pm^* + \sinh(\theta) A_\mp\right) \left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp A_\mp^* + \sinh(\theta) \cosh(\theta) \left(A_+^* A_-^* + A_+ A_-\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp^* A_\mp + \sinh^2(\theta) \left[A_\mp, A_\mp^*\right] \\ &\quad + \sinh(\theta) \cosh(\theta) \left(A_+^* A_-^* + A_+ A_-\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp^* A_\mp \\ &\quad + \sinh(\theta) \cosh(\theta) \left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta)\,.\end{aligned}$$

Using the fact that $\sinh(2\theta) = 2 \sinh(\theta) \cosh(\theta)$ and $\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(2\theta)$, as a consequence we get

$$\begin{aligned}C_+^* C_+ + C_-^* C_- &= \cosh^2(\theta) A_+^* A_+ + \sinh^2(\theta) A_-^* A_- \\ &\quad + \sinh(\theta) \cosh(\theta) \left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta) \\ &\quad + \cosh^2(\theta) A_-^* A_- + \sinh^2(\theta) A_+^* A_+ \\ &\quad + \sinh(\theta) \cosh(\theta) \left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta) \\ &= \cosh(2\theta) \left(A_+^* A_+ + A_-^* A_-\right) + \sinh(2\theta) \left(A_+^* A_-^* + A_+ A_-\right) \\ &\quad + 2 \sinh^2(\theta)\,.\end{aligned}$$

Now notice that

$$\left(\frac{\eta}{\sqrt{\eta^2 - \zeta^2}}\right)^2 - \left(\frac{\zeta}{\sqrt{\eta^2 - \zeta^2}}\right)^2 = 1.$$

As a consequence, there is θ such that $\cosh(2\theta) = \frac{\eta}{\sqrt{\eta^2 - \zeta^2}}$ and $\sinh(2\theta) = \frac{\zeta}{\sqrt{\eta^2 - \zeta^2}}$, and as a consequence

$$\begin{aligned}H &= \eta \left(A_+^* A_+ + A_-^* A_-\right) + \zeta \left(A_+^* A_-^* + A_+ A_-\right) \\ &= \sqrt{\eta^2 - \zeta^2} \left[\cosh(2\theta) \left(A_+^* A_+ + A_-^* A_-\right) + \sinh(2\theta) \left(A_+^* A_-^* + A_+ A_-\right)\right] \\ &= \sqrt{\eta^2 - \zeta^2} \left[C_+^* C_+ + C_-^* C_- - 2 \sinh^2(\theta)\right],\end{aligned}$$

so $\alpha = \sqrt{\eta^2 - \zeta^2}$. Now, we have

$$\begin{aligned}-2\sqrt{\eta^2 - \zeta^2} \sinh^2(\theta) &= \sqrt{\eta^2 - \zeta^2} (1 - \cosh(2\theta)) = \sqrt{\eta^2 - \zeta^2} \left(1 - \frac{\eta}{\sqrt{\eta^2 - \zeta^2}}\right) \\ &= \sqrt{\eta^2 - \zeta^2} - \eta = -\frac{\zeta^2}{\eta + \sqrt{\eta^2 + \zeta^2}}.\end{aligned}$$

With α as above and $\beta = \sqrt{\eta^2 - \zeta^2} - \eta$ we then get

$$\begin{aligned} H &= \sqrt{\eta^2 - \zeta^2}\left(C_+^* C_+ + C_-^* C_-\right) - \frac{\zeta^2}{\eta + \sqrt{\eta^2 + \zeta^2}} \\ &= \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta \end{aligned}$$

□

11 Exercise Sheet 11

11.1 Exercise 1 - Double Harmonic oscillator

Let $\mathcal{H} = L^2\left(\mathbb{R}^2\right)$. Let $\widetilde{H}$ be defined as

$$\widetilde{H} := -\frac{1}{2}\left(\Delta_x + \Delta_y\right) + \frac{1}{2}\left(x^2 + y^2\right) - \lambda xy \tag{71}$$

with $D\left(\widetilde{H}\right) = C_c^\infty\left(\mathbb{R}^2\right)$.

Prove that if $\lambda \in (-1, 1)$ then $\widetilde{H}$ is essentially self adjoint and study the spectrum of the closure of $\widetilde{H}$.

Hint: Prove that, with the right change of variables $(x, y) \to (w, z)$, $\widetilde{H} = H_w + H_z$ with H_w only depending on w and H_z only depending on z.

Proof. Consider the change of variables given as $z := x + y,\ w = x - y$. If we define $\phi(z, w) := \psi\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$ we get that

$$\begin{aligned}\Delta_x \psi(x, y) &= \Delta_x\left[\phi(x+y, x-y)\right] = \partial_x\left[\left(\partial_z\phi\right)(x+y, x-y) + \left(\partial_w\phi\right)(x+y, x-y)\right]\\ &= \left(\Delta_z\phi\right)(x+y, x-y) + 2\left(\partial_z\partial_w\phi\right)(x+y, x-y) + \left(\Delta_w\phi\right)(x+y, x-y)\end{aligned}$$

and analogously

$$\Delta_y\psi(x, y) = \left(\Delta_w\phi\right)(x+y, x-y) - 2\left(\partial_z\partial_w\phi\right)(x+y, x-y) + \left(\Delta_w\phi\right)(x+y, x-y).$$

Now, it is easy to check that

$$\begin{aligned} x^2 + y^2 &= \frac{z^2 + w^2}{2}\\ xy &= \frac{z^2 - y^2}{4}. \end{aligned}$$

As a consequence, we get that $\psi \in C_c^\infty\left(\mathbb{R}^2\right)$ if and only if $\phi \in C_c^\infty\left(\mathbb{R}^2\right)$ and moreover

$$\widetilde{H}\psi(x, y) = \left[-\left(\Delta_z\phi\right) + \frac{1-\lambda}{4}z^2\right]\phi(z, w) + \left[-\left(\Delta_w\phi\right) + \frac{1+\lambda}{4}w^2\right]\phi(z, w).$$

If we denote now $H_\omega := -\Delta + \omega^2\xi^2$ as the harmonic oscillator in one dimension with variable ξ, we know that H_ω is self adjoint with domain $D_\omega := \left\{f \in L^2(\mathbb{R}) \mid \xi^2 f,\ k^2\widehat{f} \in L^2(\mathbb{R})\right\}$.

As a consequence, the operator defined as $H_{\sqrt{1-\lambda}/2} \otimes \mathrm{id} + \mathrm{id} \otimes H_{\sqrt{1+\lambda}/2}$ is self-adjoint with domain[7] $D_{\sqrt{1-\lambda}/2} \otimes D_{\sqrt{1+\lambda}/2}$. Given that this operator corresponds to the closure of $\widetilde{H}$, we get that $\widetilde{H}$ is essentially self-adjoint.

[7] Recall that given two vector subspaces $V_1,\ V_2$ of $L^2(\mathbb{R})$, we have that the space $V_1 \otimes V_2$ is defined as the closure in $L^2\left(\mathbb{R}^2, dxdy\right)$ of all possible linear combinations of product of one element of V_1 with one element of V_2, i.e.

$$V_1 \otimes V_2 = \overline{\left\{\sum_{j=1}^N v_j^1(x)\, v_j^2(y) \mid v_j^1 \in V_1,\ v_j^2 \in V_2,\ \forall j = 1, \ldots, N\right\}}.$$

Now, in the exercise session we saw that

$$\sigma\left(H_{\sqrt{1-\lambda/2}}\otimes \mathrm{id}+\mathrm{id}\otimes H_{\sqrt{1+\lambda/2}}\right)=\overline{\sigma\left(H_{\sqrt{1-\lambda/2}}\right)+\sigma\left(H_{\sqrt{1+\lambda/2}}\right)},$$

and in class we saw that[8] $\sigma\left(H_\omega\right)=\omega+2\omega\mathbb{N}$, therefore we can conclude that

$$\sigma\left(\widetilde{H}^{\mathrm{cl}}\right)=\left\{\frac{\sqrt{1+\lambda}+\sqrt{1-\lambda}}{2}+\sqrt{1+\lambda}n+\sqrt{1-\lambda}m \mid n,m\in\mathbb{N}\right\}.$$

□

11.2 Exercise 2 - Normal matrices polynomials

Let A be a normal matrix (meaning that $AA^*=A^*A$) and p a polynomial in two variables. Show by example that an eigenvector for $p\left(A,A^*\right)$ is not necessarily an eigenvector for A.

Remark: Even if eigenvectors of $p\left(A,A^*\right)$ do not correspond to eigenvectors of A, the spectrum does, in the sense that

$$\sigma\left(p\left(A,A^*\right)\right)=\left\{p\left(\lambda,\lambda^*\right)\mid \lambda\in\sigma\left(A\right)\right\}. \tag{72}$$

Proof. Consider the matrix A defined as

$$A:=\begin{pmatrix}0 & 1+i\\ -1-i & 0\end{pmatrix}.$$

We can compute explicitly the adjoint matrix A^* and we get

$$A^*=\begin{pmatrix}0 & -1+i\\ 1-i & 0\end{pmatrix}=iA.$$

As a consequence we get $\left[A,A^*\right]=i\left[A,A\right]=0$ and A is therefore normal. Let now $p\left(x,y\right)=xy$. We get that $p\left(A,A^*\right)=iA^2$. Now, another explicit computation gives us that

$$p\left(A,A^*\right)=iA^2=i\begin{pmatrix}-2i & 0\\ 0 & -2i\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}=2\,\mathrm{id}\,.$$

Now, given that $p\left(A,A^*\right)=2\,\mathrm{id}$, both $e_1:=\left(1,0\right)^{\mathrm{T}}$ and $e_2:=\left(0.1\right)^{\mathrm{T}}$ are eigenvector, but $Ae_1=-\left(1+i\right)e_2$ and $Ae_2=\left(1+i\right)e_1$, which shows that this is in fact a counter example.

□

[8] Recall that $0\in\mathbb{N}$.

11.3 Exercise 3 - Spectral measure of the laplacian

Let $I := [0,1]$ and consider $\mathcal{H} = L^2(I)$. Define the operator $H := -\Delta$ with domain[9] $D(H) := H^2(I) \cap C^1_{\text{per}}(I)$. Prove that H is self-adjoint and exhibit its spectral measure explicitly.

Proof. Given that H is symmetric, we get that $D(H) \subseteq D(H^*)$. Let now $\psi \in D(H^*)$. Recall that if

$$\mathcal{F}\psi(k) = \hat{f}(k) = \int_I e^{-2\pi ikx} f(x)\, dx$$

is the Fourier series associated to ψ. We then have that

$$\begin{aligned}\mathcal{F}(\partial_x \psi)(k) &= 2\pi ik\hat{\psi}(k)\\ \mathcal{F}(\Delta_x \psi)(k), &= -(2\pi k)^2 \hat{\psi}(k).\end{aligned}$$

Also, we mentioned before the fact that the Fourier series acts as a unitary operator. Consider now the state ψ_Λ defined as

$$\psi_\Lambda(x) = \sum_{k=-\Lambda}^{\Lambda} (2\pi k)^2 \hat{\psi}(k) e^{2\pi ikx}.$$

Clearly $\hat{\psi}_\Lambda \in D(H)$. From the definition of Fourier transform we get that $\hat{\psi}_\Lambda(k) = (2\pi k)^2 \hat{\psi}(k)$ for any $|k| \leqslant \Lambda$ and $\hat{\psi}_\Lambda(k) = 0$ otherwise. Moreover we have

$$\|\psi_\Lambda\|^2_{L^2(I)} = \sum_{k=-\Lambda}^{\Lambda} (2\pi k)^4 \left|\hat{\psi}(k)\right|^2.$$

Now, we know that $|\langle H\psi_\Lambda, \psi\rangle| \leqslant C\|\psi_\Lambda\|_{L^2(I)}$, therefore

$$\begin{aligned}C\|\psi_\Lambda\|_{L^2(I)} \geqslant |\langle H\psi_\Lambda, \psi\rangle| &= \left|\sum_{k=-\Lambda}^{\Lambda} (2\pi k)^2 \overline{\hat{\psi}_\Lambda(k)}\hat{\psi}(k)\right|\\ &= \left|\sum_{k=-\Lambda}^{\Lambda} (2\pi k)^4 \left|\hat{\psi}(k)\right|^2\right| = \|\psi_\Lambda\|_{L^2(I)}.\end{aligned}$$

This implies that $\sup_{\Lambda\in\mathbb{N}} \|\psi_\Lambda\|_{L^2(I)} \leqslant C$, and therefore $\psi \in H^2(I)$. Now, consider $\varphi \in D(H)$. Integrating by part we get

$$\begin{aligned}\langle \varphi, H^*\psi\rangle = \langle H\varphi, \psi\rangle &= \int_I \overline{-\partial_x^2 \varphi(x)}\psi(x)\, dx\\ &= -\partial_x\varphi(1)[\psi(1) - \psi(0)] + \int_I \overline{\partial_x\varphi(x)}\partial_x\psi(x)\, dx\\ &= -\partial_x\varphi(1)[\psi(1) - \psi(0)] + \varphi(1)[\partial_x\psi(1) - \partial_x\psi(0)]\\ &\quad + \int_I \overline{\varphi(x)}\left(-\partial_x^2\psi(x)\right) dx.\end{aligned}$$

[9] This definition makes sense, because we know that for any function $\psi \in H^2(I)$ we have that there is a function $\tilde{\psi} \in C^1(I)$ that coincides almost everywhere with ψ. The definition of the domain is then the set of functions $\psi \in H^2(I)$ such that the function $\tilde{\psi}$ is periodic with derivative which is periodic.

Considering functions such that $\varphi(0) = \partial_x \varphi(0) = 0$, we get that $H^* \psi(x) = -\partial_x^2 \psi(x)$. As a consequence we get that for any function $\psi \in D(H)$

$$-\partial_x \varphi(1)\left[\psi(1) - \psi(0)\right] + \varphi(1)\left[\partial_x \psi(1) - \partial_x \psi(0)\right] = 0.$$

as a consequence, $\psi(1) = \psi(0)$ and $\partial_x \psi(1) = \partial_x \psi(0)$ and $\psi \in D(H)$ and therefore H is self-adjoint.

We now get that for any $\varphi \in \mathcal{H}$ and $\psi \in D(H)$

$$\langle \varphi, H\psi \rangle = \sum_{k \in \mathbb{N}} (2\pi k)^2 \, \overline{\widehat{\varphi}(k)} \widehat{\psi}(k) = \langle \varphi, \sum_{k \in \mathbb{N}} (2\pi k)^2 \, \widehat{\psi}(k) \, e^{2\pi i k x} \rangle = \langle \varphi, \sum_{k \in \mathbb{N}} (2\pi k)^2 \, P_k \psi \rangle,$$

where P_k is the projector along the function $e^{2\pi i k x}$. Therefore, given that $H = \sum_{k \in \mathbb{N}} (2\pi k)^2 \, P_k$, the spectrum of H is given by $\sigma(H) = \left\{ 4\pi^2 k^2 | \; k \in \mathbb{N} \right\}$.

We can then write H as

$$H = \sum_{\lambda \in \sigma(H)} \lambda \left[P_{\frac{\sqrt{\lambda}}{2\pi}} + P_{-\frac{\sqrt{\lambda}}{2\pi}} \right],$$

and therefore the projection-valued measure associated to H is given by

$$\mu(E) = \sum_{\lambda \in E} \lambda \left[P_{\frac{\sqrt{\lambda}}{2\pi}} + P_{-\frac{\sqrt{\lambda}}{2\pi}} \right],$$

for any E measurable subset of $\sigma(H)$.

□

11.4 Exercise 4 - Bogoliubov diagonalization - part II

Let $\mathcal{H}$ be an Hilbert space and $A_+, \; A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^* \right] = \mathrm{id}, \tag{73}$$

$$\left[A_+, A_- \right] = \left[A_+, A_-^* \right] = 0. \tag{74}$$

Let moreover $\eta, \; \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta \left(A_+^* A_+ + A_-^* A_- \right) + \zeta \left(A_+^* A_-^* + A_+ A_- \right). \tag{75}$$

Recall that if $\theta = \frac{1}{2} \operatorname{arctanh}\left(\frac{\zeta}{\eta}\right)$, $\alpha = \sqrt{\eta^2 - \zeta^2}$, $\beta = \sqrt{\eta^2 - \zeta^2} - \eta$ and C_+ and C_- are defined as

$$C_\pm := \cosh(\theta) \, A_\pm + \sinh(\theta) \, A_\mp^* \tag{76}$$

we get

$$\left[C_\pm, C_\pm^* \right] = \mathrm{id}, \tag{77}$$

$$\left[C_+, C_- \right] = \left[C_+, C_-^* \right] = 0, \tag{78}$$

$$H = \alpha \left(C_+^* C_+ + C_-^* C_- \right) + \beta. \tag{79}$$

a Consider $X := A_+^* A_-^* - A_+ A_-$. Prove that X is skew-adjoint, meaning that $X^* = -X$.

b For any $t \in \mathbb{R}$ consider $U(t) := e^{-tX}$. Prove that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group such that

$$U(t) A_\pm U(-t) = \cosh(t) A_\pm + \sinh(t) A_\mp^*. \tag{80}$$

Hint: Consider for any $\psi, \varphi \in \mathcal{H}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\pm(t) := \langle \psi, U(t) A_\pm U(-t) \varphi \rangle. \tag{81}$$

Prove that f satisfies a closed second order differential equation and deduce (80).

c Suppose that there is a complete orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that $A_\pm^* A_\pm \varphi_n = \epsilon_n^\pm \varphi_n$, with $\epsilon_n^\pm \in \mathbb{R}$. Prove that there exist a complete orthonormal system $\{\psi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that

$$H\psi_n = \left[\alpha\left(\epsilon_n^+ + \epsilon_n^-\right) + \beta\right]\psi_n. \tag{82}$$

Proof. To prove **a** is enough to notice that, given that $A_\pm$ are bounded and that $A_\pm^{**} = A_\pm$, then $X^* = \left(A_+^* A_-^* - A_+ A_-\right)^* = A_- A_+ - A_-^* A_+^* = -X$.

To prove **b**, define $Y := iX$; then the operator Y is is firstly bounded because X is, and moreover is now self adjoint, indeed $Y^* = (iX)^* = -iX^* = iX = Y$. We can then construct via functional calculus the operator $U(t) := e^{itY} \equiv e^{-tX}$ as a strongly continuous one-parameter unitary group. For any $\psi \in \mathcal{H}$ we then have, by the Stone theorem, that

$$\lim_{h\to 0} \frac{U(t+h) - U(t)}{h}\psi = iYU(t)\psi = -XU(t)\psi.$$

Now, consider $\psi, \varphi \in \mathcal{H}$. Define then $f_\pm(t)$ as in (81), we then get that, given that $U(t)$ is a strongly continuous one-parameter unitary group, f is a differentiable function and its derivative satisfies

$$\begin{aligned}
f_\pm'(t) &= \partial_t \langle U(-t)\psi, A_\pm U(-t)\varphi\rangle \\
&= \langle XU(-t)\psi, A_\pm U(-t)\varphi\rangle + \langle U(-t)\psi, A_\pm XU(-t)\varphi\rangle \\
&= -\langle \psi, U(t) XA_\pm U(-t)\varphi\rangle + \langle \psi, U(t) A_\pm XU(-t)\varphi\rangle \\
&= -\langle \psi, U(t)[X, A_\pm] U(-t)\varphi\rangle.
\end{aligned}$$

Consider now $[X, A_+]$; we get

$$[X, A_+] = \left[A_+^* A_-^* - A_+ A_-, A_+\right] = \left[A_+^* A_-^*, A_+\right] = \left[A_+^*, A_+\right] A_-^* = -A_-^*,$$

and similarly we get

$$\begin{aligned}
&[X, A_-] = -A_+ \\
&\left[X, A_+^*\right] = -\left[X^*, A_+^*\right] = [X, A_+]^* = -A_-^{**} = -A_- \\
&\left[X, A_-^*\right] = -\left[X^*, A_-^*\right] = [X, A_-]^* = -A_+^{**} = -A_+.
\end{aligned}$$

From the fact that $[X, A_\pm]$ is bounded we also have that $f'_\pm$ is again differentiable and we get

$$\begin{aligned} f''_\pm(t) &= \partial_t \langle U(-t)\psi, A^*_\mp U(-t)\varphi\rangle = -\langle \psi, U(t)\left[X, A^*_\mp\right] U(-t)\varphi\rangle \\ &= -\langle \psi, U(t)(-A_\pm) U(-t)\varphi\rangle = f(t). \end{aligned}$$

As a consequence we get that f solves the following second order ordinary differential equation

$$\begin{cases} f''_\pm = f_\pm, \\ f_\pm(0) = \langle \psi, A_\pm \varphi\rangle, \\ f'_\pm(0) = \langle \psi, A^*_\mp \varphi\rangle. \end{cases}$$

From the fact that $f''_\pm = f_\pm$, we get $f_\pm(t) = f_\pm(0)\cosh(t) + f'_\pm(0)\sinh(t)$. As a consequence we get that for any $\psi,\ \varphi \in \mathcal{H}$

$$\begin{aligned} \langle \psi, U(t) A_\pm U(-t)\varphi\rangle = f_\pm(t) &= f_\pm(0)\cosh(t) + f'_\pm(0)\sinh(t) \\ &= \langle \psi, A_\pm\varphi\rangle \cosh(t) + \langle \psi, A^*_\mp \varphi\rangle \sinh(t) \\ &= \langle \psi, \left[\cosh(t) A_\pm + \sinh(t) A^*_\mp\right]\varphi\rangle, \end{aligned}$$

and therefore we obtain (80).

To prove **c**, consider the pair of operators $C_\pm$ defined as in 76. From point **b** if we define $U := U(\theta)$ we get that

$$\begin{aligned} UA_\pm U^* &= \cosh(\theta) A_\pm + \sinh(\theta) A^*_\mp = C_\pm \\ UA^*_\pm U^* &= (UA_\pm U^*)^* = C^*_\pm. \end{aligned}$$

Using we can then rewrite the Hamiltonian as

$$H = \alpha\left(C^*_+ C_+ + C^*_- C_-\right) + \beta = U\left[\alpha\left(A^*_+ A_+ + A^*_- A_-\right) + \beta\right] U^*.$$

Define now $\psi_n := U\varphi_n$; then on the one hand we have $\langle \psi_n, \psi_m\rangle = \langle U\varphi_n, U\varphi_m\rangle = \langle \varphi_n, U^* U\varphi_m\rangle = \langle \varphi_n, \varphi_m\rangle = \delta_{n,m}$ so $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal system. Given that $\{\varphi_n\}_{n\in\mathbb{N}}$ is also complete and U is bijective, we get that also $\{\psi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system. Now we then get

$$\begin{aligned} H\psi_n &= U\left[\alpha\left(A^*_+ A_+ + A^*_- A_-\right) + \beta\right] U^*(U\varphi_n) = U\left[\alpha\left(A^*_+ A_+ + A^*_- A_-\right)\varphi_n\right] + \beta\psi_n \\ &= U\left[\alpha\left(\epsilon^+_n + \epsilon^-_n\right)\varphi_n\right] + \beta\psi_n = \left[\alpha\left(\epsilon^+_n + \epsilon^-_n\right) + \beta\right]\psi_n, \end{aligned}$$

which concludes the proof.

□